Ergo AN OPEN ACCESS JOURNAL OF PHILOSOPHY

What's So Special About Reasoning? Rationality, Belief Updating, and Internalism

WADE MUNROE
University of Michigan, Department of Philosophy and the Weinberg Institute for Cognitive Science

In updating our beliefs on the basis of our background attitudes and evidence we frequently employ objects in our environment to represent pertinent information. For example, we may write our premises and lemmas on a whiteboard to aid in a proof or move the beads of an abacus to assist in a calculation. In both cases, we generate extramental (that is, occurring outside of the mind) representational states, and, at least in the case of the abacus, we operate over these states in light of their contents (e.g., the integers represented by the beads) to generate new representations. In this paper, I argue that our belief updating processes and the grounds of their rational evaluation are partly constituted by extramental representations and operations. In other words, we don't merely update our attitudes through an internal process of reasoning on the basis of available evidence. If we are to accurately understand and rationally evaluate our belief updating processes and resultant attitudes, we need to examine how we representationally appropriate our extramental environment in the updating process.

As epistemic agents, we have the capacity to update our beliefs at the personal level. In doing so, we engage in controlled, reason-responsive operations over representational states in light of their contents (e.g., the propositional contents of our premise beliefs) performed with the end of determining the beliefs that are rationally appropriate, given our background attitudes and evidence. In updating our beliefs, we frequently employ objects in our environment to represent pertinent information. For instance, we may write our premises and lemmas on a whiteboard to aid in a proof or move the beads of an abacus to assist in calculation. In both cases, we generate extramental (that is, occurring outside

Contact: Wade Munroe <wemunroe@umich.edu>

https://doi.org/10.3998/ergo.5182

of the mind[1]) representational states, and, at least in the case of the abacus, we operate over these states in light of their contents (e.g., the integers represented by the beads) to generate new representations. In intentionally moving the beads of the abacus in accordance with rules for abacus calculation we, thereby, perform an arithmetic operation.

In this paper, I explore the epistemic role of extramental representations and our operations over them in updating our beliefs. More specifically, I argue for the following:

> **Extramental Thesis**: Our belief updating processes and the grounds of their rational evaluation are partly constituted by extramental representations and operations.[2]

The paper proceeds as follows: In §1, I clarify Extramental Thesis in the context of recent philosophical work on reasoning and inference. In §2, I provide an overview of the central argument for Extramental Thesis. In §3 and §4, I offer support for the premises of the argument. Finally, in §5, I answer objections.

1. Reasoning and Extramental Thesis

My interests in this paper concern (i) the types of representational states and state transitions that constitute our personal-level belief updating processes and (ii) the grounds of epistemic evaluation for these processes and our resultant beliefs. Philosophers typically assume that our personal-level belief updating processes are exhausted by reasoning. However, "reasoning," "inference," and cognate notions are used in multiple ways across philosophy and the psychological sciences; thus, more needs to be said to clarify the dominant philosophi-

1. I take 'mind' to designate the mental states and processes used in intentional-psychological explanations of behavior. Paradigmatic mental states and processes include beliefs, desires, inferences, and associations. Mental states can be factive—as in knowing or seeing that something is the case—as well as non-factive—as in believing that something is the case (Nagel 2013). I assume that the examples of extramental representational states and operations that I discuss, e.g., the beads of an abacus and the physical act of moving the beads, occur outside of the mind (in my sense of the term). Although the beads may represent integers and moving the beads may constitute an arithmetic operation, the beads and our operations over them aren't mental states and processes, respectively (e.g., moving the beads of an abacus isn't performing an inference over a set of mathematical beliefs).

2. Although I focus on belief updating processes in this paper, my arguments could easily be extended to cover the revision of attitudes of other types, e.g., intentions. By 'process' I just mean a set of states and state transitions or operations. Insofar as we are dealing with personal-level processes of belief updating, I assume the relevant states are representational.

cal position on i and ii.[3] The vast majority of recent philosophical work on reasoning takes (what I call) an epistemology-first approach in which the relevant notion of reasoning is fixed by a series of epistemic functional descriptions, that is, descriptions that characterize reasoning in terms of the roles that the process type—whatever it is—plays in epistemic theorizing (Boghossian 2014; 2018; 2019; Broome 2013; Valaris 2017). For example, Paul Boghossian writes,

> Well, for better or worse, in epistemology we are obsessed with the notion of rationality. We are obsessed with the idea that there are better and worse ways for you to manage your beliefs; and that these ways reflect on your virtues as an epistemic agent. . . .Reasoning in this sense is a recognizable epistemic kind. It is a personal-level establishment of your reasons for belief, in a way that grounds both the assessment of the rationality of what you believe and of your virtues as an epistemic agent. (2018: 60–61)

Boghossian notably refers to reasoning as an epistemic kind, as opposed to, say, a cognitive kind fixed by the functional roles assigned to the process type in a developed theory of cognition.[4] The relevant notion of reasoning is marked by its role in the assessment of our beliefs and epistemic virtues, as manifested in our personal-level processes of managing our attitudes.[5]

In his (2018), Boghossian discusses three epistemic roles (apparently) played by reasoning. Reasoning (according to Boghossian) is whatever type of personal-level process of managing our beliefs that

> **Basing:** establishes (inferential) epistemic basing relations and, thus, the reasons for which we hold our beliefs (where the reasons for which we hold our beliefs determine the status of our beliefs as doxastically justified or not[6]),

3. Outside of (Koziolek 2021; Quilty-Dunn & Mandelbaum 2018) contemporary philosophers rarely distinguish between reasoning and inference. I follow suit and speak of reasoning and inference interchangeably. However, it should be noted that in older philosophical discussions it appears more popular to distinguish between the two (e.g., Brown 1955; Ryle 1949; Welsh 1957).

4. The less dominant philosophical approach treats reasoning as a cognitive kind and fixes the relevant notion of reasoning in terms of various cognitive parameters, independent of considerations of establishing our reasons for belief and rational evaluation (Quilty-Dunn & Mandelbaum 2018; 2019; Warren 2022).

5. Similarly, as John Broome writes, "[w]e know that people have a particular means of coming to satisfy some of the requirements of rationality, and that is reasoning. Reasoning is something we do. It is a mental activity of ours that can bring us to satisfy some of the requirements of rationality" (2013: 3).

6. Roughly, propositional justification is a function of having good reason to adopt a belief, B—regardless of whether one actually possess B—whereas doxastic justification is a function of epistemically basing B on those reasons (Neta 2019; Silva & Oliveira 2020).

Responsibility: we can be held epistemically responsible for, and

Virtue: grounds our virtues as epistemic agents.

Although Boghossian mentions all three of the above, he claims that Basing is the 'minimal core' of reasoning. Boghossian's focus on Basing is unsurprising, given that philosophers typically analyze (inferential) epistemic basing and inference in terms of one another, thus treating inference and basing as two sides of the same coin.[7] Following Boghossian, I focus on Basing as the principal functional role that fixes the relevant type of personal-level process of managing our beliefs that "plays a central role in epistemology" (2018: 60). For the sake of space, I leave the exploration of how Extramental Thesis may interact with accounts of epistemic virtue and responsibility for future work.

The dominant view in the extant literature as developed by Boghossian (2014; 2018), John Broome (2013; 2014), and others is that the process type that plays the roles of Basing, Responsibility, and Virtue is a mental, personal-level process of arriving at or altering a propositional attitude solely by operating over further attitudes in a rule-governed fashion. "Reasoning" herein will pick out this mental, personal-level, rule-governed process. In addition, it's commonly assumed that rational reasoning consists in following the right rules.[8] As Broome puts it, "to reason correctly, you must follow a correct rule" (2014: 624). So, whether we have rationally managed our beliefs at the personal level and whether our resultant beliefs are properly epistemically based (and, thus, doxastically justified) are purely functions of internal features, namely, whether (i) the (propositional contents of the) attitudes from which we reason epistemically support our conclusions and (ii) we follow the right rules in our inferential transitions. Thus, on the dominant philosophical view, the types of representational states and state transitions that constitute our belief updating processes and ground epistemic evaluation are, respectively, propositional attitudes and inferential transitions.

I join the philosophical majority in taking an epistemology-first approach to fixing the relevant notion of a belief-updating process as whatever type of personal-level process of managing our beliefs plays the Basing role. Where

7. For example, as Robert Audi (1986: 31) writes, "[a] belief for a reason is quite naturally conceived as inferential . . ." and as Barbra Winters writes, ". . . it is reasoning from p which is required for one's belief that q to be based on p" (Winters 1980).

8. Not all philosophers accept a rule-following account of (rational) reasoning. Kathrin Glüer and Åsa Wikforss (2009; 2013), for example, argue that rationality cannot be a function of rule-following. However, their arguments rest on an overly intellectualized notion of what it is to follow a rule. There are accounts of rule-following that aren't as involved as the analysis developed by Glüer and Wikforss (e.g., Broome 2013; 2014; Toppinen 2015). A detailed discussion of rule-following is beyond the scope of this paper.

I depart from the majority is in arguing that if we are interested in the process type that plays the Basing role (as well as the Responsibility and Virtue roles, although these won't be my focus), there is no reason to restrict discussion to propositional attitudes and inferential transitions. The representational states and operations that play the Basing role include extramental states and operations. We establish our reasons for belief—and, thus, the epistemic status of our conclusion attitudes—not merely through inferential operations over propositional attitudes but also through our execution of various rule-governed operations over extramental representations. Insofar as (i) we are following the philosophical majority in fixing our personal-level belief updating processes functionally as whatever representational states and operations play the Basing role, and (ii) extramental states and operations (in part) play the Basing role, then (Extramental Thesis) our belief updating processes and the grounds of their rational evaluation are partly constituted by extramental representations and operations.

Before ending the section, a note about terminology is in order: As previously stated, I will use the terms "reasoning" and "inference" as they are traditionally employed in philosophy, namely, to pick out the mental, personal-level process of arriving at or altering a belief solely by operating over further propositional attitudes in a rule-governed fashion. My expression, "belief updating process," should be understood to designate the epistemically central, rationally evaluable process of operating over representational states (in light of their contents) in a rule-governed fashion in arriving at or altering a belief that plays the Basing role, where these representational states and operations shouldn't antecedently be assumed to be mental. According to the dominant philosophical view, our belief updating processes are exhausted by reasoning, qua purely mental process. On my view, our belief updating processes extend beyond reasoning, and beyond the bounds of the mental, to include extramental representations and our operations over them.[9]

---

9. Even on process reliabilist conceptions of justification (e.g., Bergmann 2006; Goldman 1979; 1986), agent reliabilist conceptions of knowledge (e.g., Greco 1999; Greco 2003), and safety/sensitivity accounts of knowledge, which are commonly relativized to belief formation methods (Broncano-Berrocal 2014; Pritchard 2008), it is frequently assumed that our belief updating processes are purely mental. The view I develop in this paper should be contrasted with, say, the process reliabilism of someone like Alvin Goldman (e.g., 1979). For Goldman, our belief updating processes are mental; however, the features that ground the epistemic evaluation of the processes are (in part) external to the processes and the agents who use them. On my view, our belief updating processes can be partially constituted by extramental representational states and operations, but the grounds of the rational evaluation of these processes and resultant beliefs need not be external to the processes themselves, although they will be external to the minds of the epistemic agents.

2. Central Argument

Before presenting the central argument for Extramental Thesis, some additional introductory remarks and discussion of terminology are in order. Throughout the paper, my examples of belief updating processes will consist of arithmetic calculations. Mental calculation (that is, calculating internally without the aid of extramental tools) is a cognitively demanding task that loads on working memory (LeFevre, DeStefano, Coleman, & Shanahan 2005).[10] Mental calculation requires conscious, attentional resources to utilize a learned algorithm to token a series of representations of arithmetic expressions and to operate over these representations in a rule-governed manner. For instance, a person may mentally utilize long multiplication (LM) in coming to a belief about the product of 15 and 6. Using LM, like any calculation algorithm, involves performing a sequence of suboperations that are basic at the personal level,[11] for example, determining the partial products of the operands and calculating their sum. In aptly, mentally using LM to determine the product of 15 and 6, a person will make a series of occurrent judgments with the respective contents (i) that the product of 6 and 5 is 30, (ii) that the product of 6 and 10 is 60, and, therefore, (iii) that the sum of the partial products is 90, which makes the product of 15 and 6 equal to 90. I take mental calculation to be a clear example of reasoning in the philosophically dominant sense of the term.

The following terminology will be used throughout the paper and will assist in economically formulating my argument,

> **Numeric code**: Numeric codes—like English number words, Arabic numerals, or the Japanese soroban abacus system (which we discuss below)—are generative, public symbol systems that include primitive numeric expressions and compositional rules for combining the primitives to generate expressions that denote successive integers. In calling a symbol system 'public' I mean that token expressions of the system can occur extramentally, for example, as spoken utterances, written inscriptions, manual signs, bead configurations, etc., and can be used to transmit information between users of the system.[12] An expression of a public symbol system, as I am using the term, is either a basic unit of meaning for the system—for example, a word of a language or a bead of an abacus—or a meaningful combination

10. Working memory is "a mental workspace that is involved in controlling, regulating, and actively maintaining relevant information" (Raghubar, Barnes, & Hecht 2010: 110).

11. An operation is basic at the personal level if one performs the operation but not by means of performing any other operation at the personal level.

12. In contrast, something like the Language of Thought is not a public symbol system, as token expressions do not exist extramentally and are not used as a medium of communicating between individuals.

of basic units—for example, a sentence or a set of beads that designates a number via the compositional rules of the abacus numeric code.

Abacus System: An abacus system is the (i) numeric code and (ii) rules for performing arithmetic operations over expressions of the code associated with a (type of) abacus, for example, the Japanese soroban or the Chinese suanpan abacus.

Because abacus systems feature centrally in my argument, I briefly demonstrate (with the aid of Figure 1) how one abacus system—the Japanese soroban abacus system (SAS)—represents numbers.

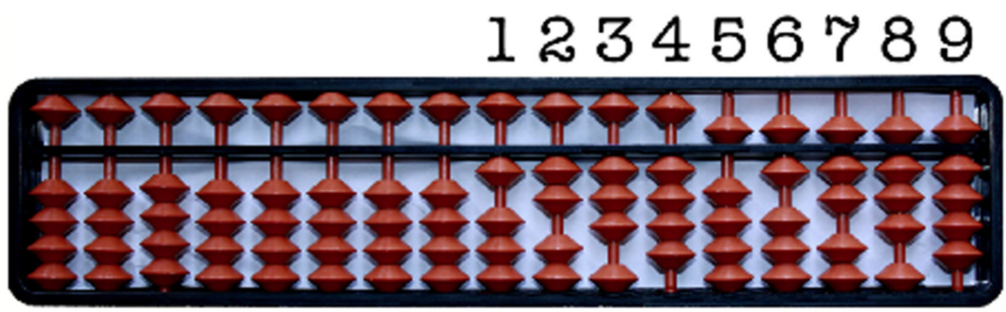

Figure 1: A Japanese soroban style abacus denoting the number 123,456,789 (from Barner et al. 2016: 2).

SAS is a base-ten positional system. The far-right column of beads is the units column, the second to right is the decades column, and so on. The beads within a column count towards the value represented by the abacus when pushed towards the bar that divides the upper bead from the four lower beads. The upper bead of a given column (when pushed towards the dividing bar) represents five multiplied by the place-value of the column, whereas the lower beads each represent one multiplied by the place-value. So, for instance, the upper bead of the third column from the right represents 500, whereas any of the lower beads represent 100. The value represented by any column is the sum of the respective values represented by the beads of the column that are pushed towards the dividing bar. Thus, the third column from the right represents 700, as the upper bead is pushed towards the center bar as well as two of the lower beads.

It's important to note that in using an abacus system in calculation certain arithmetic operations are performed by moving the beads of the abacus. In other words, the abacus is not merely a convenient mnemonic device for storing the results of calculations that happen outside of the abacus (e.g., in the mind of the abacus user). Instead, the abacus is used to perform calculations by moving the beads in accordance with the rules of the abacus system. For example, in determining the product of two numbers with SAS, the addition of the partial products happens through moving the beads. For the sake of space, I won't detail how calculation works in SAS. The interested reader may refer to (Stigler 1984).

Inner Speech: People frequently report that (at times) they experience conscious thinking as if they were 'talking in their head', thus using a natural language to verbalize their premises, lemmas, and conclusions (Heavey & Hurlburt 2008). Inner speech is the "subjective experience of language in the absence of overt and audible articulation" (Alderson-Day & Fernyhough 2015: 931). I discuss the nature of inner speech in more detail in the following section.

Mental Abacus (MA): MA is a notably efficient and effective method of mental calculation (commonly taught in certain Asian countries) that consists in visuospatially imagining a physical abacus (PA) and operating over one's imagery in a manner isomorphic to how one would operate over PA (Frank & Barner 2012). I also discuss the nature of MA in more detail in the following section.

With the above terminology in place, I now present the central argument for Extramental Thesis,

1. Public symbol systems can be used internally (solely in the mind) to various cognitive ends. For instance, one can internally use a natural language and inner speech or an abacus system and MA to express occurrent arithmetic judgments, like that the product of 6 and 5 is 30, in the process of mental calculation. (Premise supported in §3.1)
2. Given limits on our innate cognitive resources for representing exact numbers and quantities, it is necessary to mentally use a numeric code (as described in 1) in working memory as the vehicle of representation for mental calculation. In other words, in using, for example, an abacus system and MA in calculation, the MA user's reasoning—that is, their occurrent judgments, suppositions, and inferential transitions—must occur through mentally tokening expressions in the abacus system and operating over these expressions in accordance with the rules of the system. (Premise supported in §3.2)
3. Thus, in using, for example, an abacus system and MA in calculation, token expressions of the abacus system and the MA user's operations over these expressions constitute, respectively, the personal-level representa-

tional states and state transitions that (Basing) establish the MA user's reasons for belief and ground the epistemic evaluation of the MA user's conclusion attitudes. (From 1 and 2)

4. The use of MA to perform a calculation, C, involves a visuospatial simulation of the use of PA to perform C such that there is an isomorphism between MA and PA: there is a one-to-one mapping from the set of mentally realized expressions of the abacus system and operations over these expressions in using MA to the set of extramentally realized expressions (i.e., the beads of the abacus) and operations (i.e., moving the beads in accordance with the rules of the abacus system) in using PA. Using either MA or PA to perform C involves tokening and operating over the same (types of) representations—namely, expressions in the abacus system—using the same rules. (Premise supported in §4.1)
5. If (i) token expressions of the abacus system and the MA user's operations over these expressions play the Basing role in the MA user's performance of C and (ii) there exists an isomorphism between MA and PA as described in 4, then extramental token expressions of the abacus system and the PA user's operations over these expressions must also play the Basing role in the PA user's performance of C. (Premise supported in §4.2)
6. Thus, in using an abacus system and PA in calculation, part of the personal-level activity that (Basing) establishes the PA user's reasons for belief and grounds the epistemic evaluation of their resultant attitudes involves extramental expressions of the abacus system and the PA user's operations over these expressions. (From 3, 4, and 5)
7. We've fixed our personal-level belief updating processes functionally as whatever representational states and operations play the Basing role. (Stipulation discussed in §1)
8. Thus, (Extramental Thesis) our belief updating processes and the grounds of their rational evaluation are partly constituted by extramental representations and operations. (From 6 and 7)

For ease of presentation, I've formulated much of my argument in terms of MA and PA. However, as I discuss in §4.2, using PA in calculation is just one of myriad examples of how extramental representations and our operations over them can partially constitute our belief updating processes. I focus on MA and PA because I take PA to be an especially compelling instance of the broader phenomenon captured by Extramental Thesis.

I take the inferential steps (i) from 1 and 2 to 3, (ii) from 3, 4, and 5 to 6, and (iii) from 6 and 7 to 8 to be straightforward. The premises of the argument, however, clearly need to be defended. I establish the premises in the corresponding sections listed above. As I discuss in §5, my conclusion is clearly incompatible with an epistemic internalism about doxastic justification and the rationality of our belief updating processes. Internalists will likely highlight that, in using PA, our access to and operations over the abacus are mediated by perception. The obvious response to my position is to argue that the belief updating processes involved in using PA extend only so far as the PA user's perceptual access to the abacus, as opposed to being partially constituted by the abacus and the PA user's operations over the beads. In other words, the abacus and the PA user's operations over the beads don't (in part) play the Basing role; instead, it's the PA user's internal process of reasoning from her attitudes about the abacus (grounded in her perceptions of the abacus) and background mathematical attitudes that ultimately establishes the epistemic basis of her conclusion. In §5, I defend my argument against the internalist and provide an account of the epistemic role of perception in the use of PA that is consistent with Extramental Thesis.

3. Mental Calculation and Numeric Codes

In §3.1, I defend 1 of the central argument, and in §3.2, I defend 2.

3.1. Internal Use of Public Symbol Systems

Although public symbol systems, like numeric codes, may initially develop as a cultural means of representing and transmitting relevant information, they can also be used internally to consciously express our thoughts and can be put towards various cognitive ends.[13] The use of natural language in inner speech, for example, plays important roles in supporting various executive functions, like planning, inhibition, task-switching, and, more generally, reasoning (in the philosophical sense of the term) (Alderson-Day & Fernyhough 2015; Perrone-Bertolotti, Rapin, Lachaux, Baciu, & Lœvenbruck 2014). For instance, take the famous Tower of London problem, a test of planning ability used in clinical and experimental psychology in which you are presented with colored disks stacked on a series of three pegs. You must move the disks one at a time until they match another series of disks that is designated as the goal state. People typically solve the Tower of London problem with the aid of a natural language through making a series of inner speech utterances that express their sequence of planned moves (Lidstone, Meins, & Fernyhough 2010). Similarly, abacus systems can be used internally to perform arithmetic calculations through MA (Frank & Barner

13. Numeric codes originally developed in response to certain societal exigencies, like the need for various accounting practices. We humans don't spontaneously generate numeric codes in isolation of particular social needs (Cajori 1993; O'Shaughnessy, Gibson, & Piantadosi 2021).

2012). When MA users are presented with arithmetic expressions—regardless of the numeric code used to denote the expressions, for example, Japanese number words, Arabic numerals, etc.—they encode the relevant expressions into a visuospatially imaged abacus (Hishitani 1990). MA users then perform a series of operations defined over their visuospatial imagery that are isomorphic to the operations they would perform over PA.

Although it's beyond the scope of this paper to discuss cognitive models of inner speech and MA (Brooks, Barner, Frank, & Goldin-Meadow 2018; Grandchamp et al. 2019; Pickering & Garrod 2013), it's important to note that we produce inner speech and MA the same way, respectively, that we speak and operate over an actual abacus. For instance, inner speech is the result of inhibiting the process of speech planning and production prior to overt articulation, thus suppressing the execution of a planned message by our speech apparatus (that is, our vocal tract, tongue, lips, etc.). Although the planned message isn't overtly spoken, a copy of the plan (an "efference copy") and a forward model of the speech apparatus produce an internal, sensory prediction of the sound of the aborted utterance of which we become consciously aware. This sensory forward model corresponds to the 'talking-in-the-head' phenomenology of inner speech (Tian & Poeppel 2012; Whitford et al. 2017). Inner speech constitutes (in part) an offline, sensory simulation of speaking that we generate by internally using our natural language competence.

Similarly, MA is the result of an efference copy of a planned bead movement and a forward model of the motor commands involved in using one's hands with PA (Brooks et al. 2018; Wolpert, Ghahramani, & Jordan 1995). In fact, MA users frequently move their fingers and hands while mentally calculating with MA, where these movements mimic the movements they would have performed if they were using PA (Brooks et al. 2018). Just as inner speech constitutes, in part, an offline, sensory simulation of speaking, MA constitutes, in part, an offline, sensory simulation of performing a calculation through PA that the MA user generates by internally using her competence with an abacus system.

In sum, public symbol systems are not merely employed for transmitting information between users of the systems. Public symbol systems can also be used internally for various cognitive ends, including mental calculation. Thus, I've vindicated 1.

3.2. *Mental Calculation*

Despite our discussion in the previous section, we've yet to establish exactly what role numeric codes play in mental calculation. In this section, I argue that expressions in numeric codes serve as the vehicle of representation for mental calculation. In other words, the internal use of numeric codes is not merely a useful tool to foment the 'real' process of mental calculation which takes place in some other format of representation or through some other cognitive means. Mental calculation occurs through internally tokening expressions in a numeric code and operating over these expressions. I begin with a discussion of our innate cognitive resources for representing numbers before examining the role of numeric codes in mental calculation.[14]

There is a growing consensus that humans and certain non-human animals share an innate number sense (INS) that includes (i) the ability to represent exact quantity up to about three or four and (ii) an analogue representation system for dealing with approximate quantity (Feigenson, Dehaene, & Spelke 2004). The analogue representation system—commonly known as the Approximate Number System (ANS)—grounds an animal's ability to estimate and represent approximate number. So, for instance, ANS undergirds an animal's ability to represent the approximate number of animals in a pack or pieces of fruit in a bush, which can inform decisions regarding whether to engage a hostile group of conspecifics (Wilson, Hauser, & Wrangham 2001) or where to forage most efficiently (Beran 2001).

However, INS isn't sufficient to subserve exact calculation (at least when dealing with quantities larger than three or four) or to precisely discriminate between the cardinality of sets (Brannon, Jordan, & Jones 2010). Without a numeric code, we humans will have to use INS to subserve our mathematical thought, and, thus, we will lack the ability to think about or operate over numbers larger than, roughly, three or four. For example, the Pirahã and Mundurukú—two relatively isolated groups of individuals in the Amazon—use languages with very limited number vocabularies, if the languages have words for exact quantities at all. The Pirahã and Mundurukú cannot reliably perform elementary calculations or remember the cardinality of sets, despite possessing similar abilities to reason with approximate quantity as aged-matched individuals who utilize a numeric code (Frank, Everett, Fedorenko, & Gibson 2008; Gordon 2004; Pica, Lemer, Izard, & Dehaene 2004).

Similar studies have been performed with congenitally deaf homesigners raised in numerate communities. Homesigners do not acquire a conventional signed language, typically, because of a lack of access to educational resources, and thus are forced to develop a gestural idiolect (called a home sign) as a means of communicating. As Spaepen et al. report, homesigners,

> appreciate that a set of objects has an exact cardinal value, and that a unique gesture communicates that value. . . . However, despite the fact

14. The argument I provide in this section is a truncated version of an argument I've developed over several papers. For a more in-depth defense of the claims in this section see (Munroe 2022; 2023a; 2023b).

> that homesigners use their fingers to communicate about number, they do not consistently produce gestures that accurately represent the cardinal values of sets containing more than three items. (Spaepen, Coppola, Spelke, Carey, & Goldin-Meadow 2011: 3167)

If we are not taught to use a numeric code—even if we live in a numerate community in which number suffuses operations of the community, for example, in the exchange of money and goods—we will lack the means to reliably think of or operate over exact quantities.

It's important to note that becoming competent with a numeric code doesn't involve developing exact number concepts that we can use in isolation of the code. If numerate individuals are prevented from using a numeric code in calculation in virtue of an interference task or brain damage (examples of which we presently discuss), then they will have to revert to using INS and will, thus, lack the ability to perform even basic arithmetic operations. For instance, verbal shadowing—an interference task that involves repeating meaningful speech directly after hearing it, thus loading psycholinguistic processes responsible for speech planning, production, and (importantly) inner speech—severely impairs arithmetic abilities in numerate individuals who use a natural language and inner speech in calculation. During verbal shadowing, numerate individuals perform just like the Pirahã on the same tasks used to test the Pirahã's numeracy (Frank, Fedorenko, & Gibson 2008).

Unlike those who use inner speech in calculation, MA users are relatively unaffected by verbal interference tasks (Frank & Barner 2012; Hatano, Miyake, & Binks 1977). However, MA users (unlike those who use inner speech) are disrupted by motor interference tasks (Brooks et al. 2018). For example, MA users struggle to perform calculations while concurrently tapping a keyboard in a rhythmic pattern, thus loading motor planning processes that are important for generating the visuospatial imagery involved in MA (as discussed in the previous section).

Similar calculation impairments are observed in those suffering from brain damage that affects neural regions that subserve our ability to use public symbol systems. For instance, there is an extensive literature on the relation between aphasia (acquired language deficits) and acalculia (acquired mathematical deficits). It is consistently found that the severity of mathematical impairments associated with aphasia vary with the severity of the aphasia (Ardila & Rosselli 2002; Delazer & Bartha 2001; Delazer, Girelli, Semenza, & Denes 1999). For example, persons with global aphasia—the most severe form of aphasia in which one may only be able to understand or produce a handful of linguistic expressions—present the most severe mathematical deficits.

Similarly, Tanaka et al. (2012) describe a case of an expert MA user who, in virtue of a right hemispheric lesion, lost the ability to visuospatially image an abacus, and, thereby ceased to be able to perform MA. As the MA user claimed, "I lost my abacus in the brain." However, she retained an ability to use the Japanese language (which clearly includes a numeric code) and inner speech to think about exact numbers and perform calculations. As she recovered from her stroke and regained the ability to visuospatially image an abacus, fMRI revealed that the brain activity that subserved her mental calculations shifted from language areas, for example, Broca's area, to visuospatial areas, for example, the left superior partial lobule.

Moving beyond consciously thinking about numbers in working memory, there is good evidence to suggest that our arithmetic knowledge stored in long-term memory is tied to the numeric code in which we acquired the knowledge. For instance, inner speech is typically used in mental multiplication because the execution of common calculation algorithms, like LM, involves utilizing multiplication tables, which are normally acquired through rote verbal memorization (Lee & Kang 2002). In addition, bilinguals tend to be more reliable and quicker when performing mental arithmetic with the aid of the language in which they were taught basic arithmetic facts and decision procedures (Shanon 1984; Van Rinsveld, Dricot, Guillaume, Rossion, & Schiltz 2017). Despite being fluent with a second language (and being able to use that second language in inner speech, De Guerrero 2005) bilinguals translate into the language in which they received their mathematical education in order to solve arithmetic problems presented in other languages with which they are competent.

To summarize: the means we have of representing and thinking about numbers prior to acquiring a numeric code, that is, INS, is insufficient to think about or operate over numbers that exceed, roughly, three or four. However, becoming competent with a numeric code doesn't involve developing exact number concepts that we can use in isolation of the code to (i) think about or operate over exact numbers or quantities or (ii) store arithmetic facts (e.g., multiplication tables) in long-term memory. The vehicle of representation used in mental calculation is not some type of conceptual medium divorced from public symbol systems, like the Language of Thought, but expressions in a numeric code in which we've received mathematical training. Thus, the representations we generate and operate over in performing mental calculations are token expressions of a numeric code, where we generate these expressions in much the same way that we would have if we were, for example, speaking out loud or operating over an actual abacus.

My claims in the previous paragraph assume that when we internally use a public symbol system to express our thoughts through inner speech or MA we internally token expressions of that system. For the sake of space, I won't defend the assumption here, especially because the assumption has been adequately defended elsewhere (see Gauker 2018; Gregory 2016) and is generally accepted in recent work on the metaphysics of linguistic expressions. As J. T. M.

Miller notes (2020), regardless of one's theory of word and sentence types, there is general agreement that word and sentence tokens include both extramental tokens, for example, spoken, signed, and written words, and mental tokens, for example, inner speech utterances. More generally, expressions in public symbol systems aren't metaphysically tethered to any particular medium of realization, for example, sound waves, ink splotches, fingers and hands, neural states, etc. Just as words, sentences, and other linguistic expressions can occur in the mind, so can expressions in an abacus system.

An example will help to illustrate what we've established thus far. In determining the product of 15 and 6, an MA user (let's call him Tetsuo) will first determine the partial products of the operands. The vehicle of representation for Tetsuo's occurrent judgments about the partial products, for example, that the product of 5 and 6 is 30, will have to be expressions in an abacus system, given our innate cognitive limitations in representing numbers and Tetsuo's choice of numeric code. Additionally, Tetsuo will sum the partial products to arrive at his conclusion through mentally 'moving' the beads of his visuospatially imaged abacus. In order for Tetsuo to update his attitudes rationally and to properly base his belief that the product of 15 and 6 is 90 in his judgments about the partial products and their sum, he has to correctly and competently determine and sum the partial products through his operations over token expressions in the abacus system. Thus, in using an abacus system and MA in calculation, token expressions of the system and Tetsuo's operations over these expressions constitute, respectively, the personal-level representational states and state transitions that (Basing) establish Tetsuo's reasons for belief and ground the epistemic evaluation of his resultant attitudes. I've, thus, vindicated 2 of the central argument and the inference to 3.

4. MA, PA, and Basing

In §4.1, I defend 4 of the central argument, and in §4.2, I defend 5.

4.1. MA and PA Isomorphism

Given my discussion in §3.1 and §3.2 regarding the nature of MA—especially the fact that MA constitutes (in part) an offline, sensory simulation of performing PA in which the MA user internally tokens expressions in an abacus system—we have good reason to believe that the use of MA to perform a calculation, C, is isomorphic to the use of PA to perform C in that there is a one-to-one mapping from the set of mentally realized expressions of the abacus system and operations over the expressions in using MA to the set of extramentally realized expressions and operations in using PA. In further support of the isomorphism, MA users tend to make the same distribution of errors in mental calculation that PA users make in calculating with an actual abacus, while those who use, for example, inner speech and English number words tend to make different types of errors (Stigler 1984). In addition, MA users are able to report—without consulting PA—the intermediate states that the beads of an abacus need to travel through to perform a given mathematical operation, thus suggesting that the "mentally imagined abacus [of the MA user] appears to pass through the same intermediate states as the actual abacus would on the same problem" (Stigler 1984: 173).

It's important to note that in using PA to perform a calculation, C, one does not also concurrently perform C through MA (or through any other means of mental calculation for that matter). In fact, the ability to use MA in calculation developmentally follows the ability to use PA. It's only after acquiring a sufficient level of competence in using PA that one is able to simulate the motor procedures that are learned during PA training in order to use MA (Barner et al. 2016).[15] So, imagine that, in determining the product of 15 and 6, an individual (let's call her Kaori) employs PA and the same abacus system as Tetsuo. Kaori will use PA to store the partial products she calculates, as opposed to having to store those products in visuospatial working memory like Tetsuo. In addition, Kaori will sum the partial products through physically moving the beads as opposed to mentally 'moving' bead representations like Tetsuo. Nonetheless, given the isomorphism between PA and MA, Tetsuo and Kaori token representations of the same type (expressions in an abacus system) and perform the same operations over the representations in determining the product of 15 and 6. Thus, I've vindicated 4. Using either MA or PA to perform a calculation with an abacus system, S, involves tokening and operating over the same (types of) representations—namely, expressions in S—using the rules of S.

4.2. Basing and PA

In this section, I defend 5 of the central argument by using philosophical work on the epistemic basing relation to argue that the PA user's abacus and her operations over the beads are part of the personal-level activity that (Basing) establishes the PA user's reasons for belief and grounds the epistemic evaluation of her resultant attitudes.

The dominant view in the extant philosophical literature is that the epistemic

15. Similarly, children consistently employ inner speech only around six or seven years of age, well after they've begun using a natural language in overt speech (Flavell, Green, Flavell, & Grossman 1997; Geva & Fernyhough 2019). More generally, the ability to use a public symbol system internally requires an antecedent competence with overtly using the system.

basing relation is a causal relation. A set of reasons are the reasons for which an agent believes a proposition insofar as those reasons (or—if we understand reasons to be propositions—one's attitudes that express those reasons[16]) cause (or causally sustain) the belief in the right way (Korcz 1997; 2000). The 'in the right way' qualifier acknowledges the problem of deviant causal chains. Not just any causal relation between one's reasons and conclusion suffices to establish that one believes the conclusion on the basis of those reasons. In addition, many philosophers (e.g., Neta 2019; Sylvan 2016; Sylvan & Lord in press), myself included, agree with Ernest Sosa (2015) that the 'in the right way' qualifier for causal accounts of basing ought to be understood in terms of the believing agent manifesting certain types of dispositions in her updating processes, namely, competences.[17] Although, like Sosa (2015), I take the notion of a competence to be a primitive, we can roughly characterize a competence as, "a disposition of an agent to perform [a certain activity] well" (Sosa 2010: 465). So, in order for Tetsuo to properly base his belief that the product of 15 and 6 is 90 in his judgments about the partial products and their sum, it isn't sufficient that his judgments merely cause his conclusion attitude. Tetsuo's judgements must cause the attitude—and must themselves be caused—in the right way, that is, in a manner that manifests Tetsuo's competence with arithmetic calculation.

Depending on the activity, our manifestation of a competence can occur through (in part) our interactions with the extramental environment. To use an example from Sosa (2015), an archer manifests her competence to hit her targets (in part) through how she nocks her arrows and draws back the string of her bow. As discussed in §4.1, Kaori performs the same calculations using the same numeric code, expression types, and calculation algorithm as Tetsuo. Like Tetsuo, in order for Kaori to properly base her belief that the product of 15 and 6 is 90 in her judgments about the partial products and their sum, Kaori's judgements must cause her belief—and must themselves be caused—in the right way, that is, in a manner that manifests her competence with arithmetic calculation. Thus, like Tetsuo, for Kaori to update her attitudes rationally and properly base her belief, she must correctly and competently determine and sum the partial products of her operands. However, Kaori does not sum the partial products mentally like Tetsuo. Instead, Kaori sums the partial products by moving the beads

16. My arguments should be read as neutral between different accounts of the nature of reasons, e.g., reasons for belief are propositions (Dougherty 2011; Neta 2008; Williamson 2000), facts (Dancy 2000), or mental states (Conee & Feldman 2004; 2008; Mitova 2015). Regardless of one's views about the nature of reasons, one can accept that we establish a set of reasons as the reason for which we believe a proposition, in part, through tokening and operating over extramental representational states.

17. For the sake of space, I won't rehearse Sosa's arguments here. Nonetheless, Sosa has convincingly argued that casual accounts of basing need to be supplemented with an appeal to the manifestation of certain dispositions.

of an actual abacus in accordance with the rules of the abacus system. What justifies our acceptance of 5 of the central argument—thus seeing Tetsuo and Kaori as of an epistemic par in terms of how they establish their reasons for belief—is the fact that Kaori's judgments about the partial products and their sum cause her conclusion attitude (in the right way) through a process that extends outside of Kaori's mind to include the states of an actual abacus and Kaori's operations over these states that manifest her arithmetic competence.

Note how Kaori's use of PA is unlike, say, the use of a digital calculator in terms of its epistemic functional role. Although using a calculator may require that you press certain buttons to define the mathematical operations you want performed, you do not, thereby, perform the operations. In using a calculator to determine the product of 15 and 6, you do not base your conclusion in information about the partial products of the operands and their sum, as no representations of the partial products are causally connected to the adoption of your conclusion in a manner that manifests your arithmetic competence. Instead, you base your conclusion (in part) on information about what the calculator displays. In using PA to determine the product of 15 and 6, on the other hand, Kaori bases her conclusion in information about the partial products of the operands and their sum, just as Tetsuo does in using MA, in virtue of the fact that she determines and sums those partial products through moving the beads, thus manifesting her arithmetic competence in her extramental operations.

A pair of analogous cases involving Tetsuo and Kaori will provide further support for premise 5 and help to illustrate that, in order for Kaori's conclusion belief to be caused in the right way by her judgments about the partial products and their sum, Kaori must competently determine and sum the partial products through her operations over the beads. Let's first imagine that, in determining the product of 75 and 23 using MA, Tetsuo incorrectly performs an operation over his visuospatially imaged abacus in an attempt to sum a subset of the partial products. However, after shifting focal attention away from summing this subset and back to determining the other partial products of the operands, the imagined beads are 'shifted' into a position in which they display the correct sum in virtue of a chance failure of visuospatial working memory to preserve the representations that Tetsuo attempts to keep active. As Tyler Burge rightly notes, "[a]ny reasoning in time must rely on memory . . . reasoning processes' working properly depends on memory's preserving the results of previous reasoning" (Burge 1993: 463). Although it's beyond the scope of this paper to discuss cognitive models of working memory (e.g., Baddeley 2007; Baddeley 2010; 2017), keeping previous reasoning results active in visuospatial working memory, while performing other inferences and operations, requires generating and maintaining a representation of those reasoning results in a temporary cognitive buffer, what Allen Baddeley (2010) calls the 'visuospatial sketchpad'. There are documented cases in the cognitive psychological literature of failures of working memory that result in individuals misidentifying the representations they attempt to keep active in this cognitive buffer in calculation.[18] Thus, Tetsuo's case is an empirically realistic one.

Let's further imagine that when Tetsuo brings the visuospatially imaged beads displaying the running sum of the partial products back to the focal point of attention, he proceeds to perform the correct operations and ends up rightly believing that the product of 75 and 23 is 1,725. I take it to be clear that Tetsuo has not rationally updated his attitudes. The issue is that (despite arriving at the correct conclusion) Tetsuo does not accurately sum the partial products over which he operates and, thus, does not manifest his arithmetic competence. Again, in order for Tetsuo to update his attitudes rationally and to properly base his belief that the product of 75 and 23 is 1,725 in his judgments about the partial products, he has to correctly and competently perform the suboperations that are part of the calculation algorithm he uses.

Now imagine that Kaori attempts to determine the same product using PA. Kaori incorrectly performs an operation over the actual abacus in an attempt to determine the same sum of a subset of the partial products. However, when Kaori isn't looking, a chance gust of wind pushes the beads into a position in which they display the correct sum. Kaori proceeds to perform the correct operations over the beads and ends up rightly believing that the product of 75 and 23 is 1,725. I also take it to be clear that Kaori has not rationally updated her attitudes and, thus, is not doxastically justified in believing her conclusion. Again, the issue is that (despite arriving at the correct conclusion) Kaori does not accurately and competently sum the partial products over which she operates and, thus, does not manifest her calculation competence. However, Kaori's summing of the partial products happens extramentally. It's crucial that we keep the isomorphism between MA and PA in mind. In using PA to calculate the sum of her partial products, Kaori is not concurrently calculating the sum mentally; she calculates by actually moving the beads. We could even imagine that Kaori's performance error is due to a misfiring of motor signals at the subpersonal level that drive the movements of her hands without changing our judgment about the case.[19] Kaori's error is not an instance of poor mental calculation but of poor extramental calculation. The bottom line is that Kaori has not accurately summed her partial products and, thus, has not manifested her calculation competence in her belief updating process. As Kaori's case demonstrates, our epistemic competences—like the archer's competence in hitting her targets—are manifested, in

18. As I argue in Munroe (2022), the case study described in Benson and Denckla (1969) is one such instance. However, a protracted discussion of the case is beyond the scope of this paper.

19. Similarly, if Tetsuo's performance error was due to a misfiring at the subpersonal level, this wouldn't change our judgment about the case. The cases would still be analogous.

part, in our extramental activity.[20]

I've, thus, vindicated 5 of the central argument and the inference to 6. The representational states and state transitions that (Basing) establish Kaori's reasons for belief and ground the epistemic evaluation of her resultant attitudes include extramental representational states and rule-governed operations over these states. Given that we've fixed our personal-level belief updating processes functionally as whatever representational states and operations play the Basing role, Extramental Thesis follows.

Before ending the section, it's important to note that using PA in calculation is just one of myriad examples of how our operations over extramental representations can play the Basing role and, thus, partially constitute our belief updating processes. As another example—in keeping with our theme of arithmetic calculation—take finger reckoning, which is a class of methods for performing arithmetic operations with one's hands.[21] A finger reckoning system involves (i) a finger reckoning numeric code that consists in generative rules for representing successive integers on one's fingers—thus, various finger positions constitute expressions in the numeric code—and (ii) rules for performing arithmetic operations over one's digits in a manner sensitive to their numeric content. In finger reckoning, the relative positions of one's fingers serve to, say, store partial products, and one's finger movements constitute one's performing addition over these partial products. Like the PA user, the finger reckoner does not concurrently perform all relevant calculations mentally, thereby merely using her fingers to store intermediate calculations. The finger reckoner performs certain calculations through moving her fingers and, thus, (in part) manifests her arithmetic competence through her finger movements. The belief updating processes

20. The dispositions philosophers appeal to in addressing the problem of deviant causal chains for causal accounts of basing don't always make it clear how the problem can be addressed once we allow that extramental representational states and operations are (in part) playing the Basing role and, thus, properly part of the process of establishing one's reasons for belief. For example, John Turri argues that, "R is among your reasons for believing Q if and only if R's causing your belief manifests (at least some of) your cognitive traits" (2011: 393). Although Turri doesn't provide an extensive analysis of what cognitive traits are, he gives the reader a non-exhaustive list, e.g., we reason in 'patterns' that correspond to formal updating rules. It might be thought that the manifestation of our cognitive traits happens purely in the mind, and, thus, accounts like Turri's won't be of much help in determining when the extramental aspects of one's belief updating processes cause the adoption of one's conclusion 'in the right way' to establish an epistemic basing relation.

However, as I've argued, once we allow that our belief updating processes aren't solely mental, we can understand epistemic competences on par with an archer's competence in hitting targets. Like the archer's competence, our epistemic competences are manifested (in part) in our extramental activity. For example, Kaori manifests her competence in calculating through (in part) moving the beads of an abacus in 'patterns' that correspond to the formal rules of an abacus system.

21. Like the abacus, finger reckoning has a long history of use in multiple cultures (Williams & Williams 1995). The Trachtenberg and Chisanbop systems are two recently developed systems of finger reckoning (Knifong & Burton 1979).

of the reckoner, like the PA user, are (in part) constituted by a series of operations defined over extramental representations.

5. Internalism and the Epistemic Role of Perception

My position is incompatible with an epistemic internalism about doxastic justification and the rationality of our belief updating processes, where I take internalism about an epistemic property, E (e.g., the property of being rational), holding of some state or process, s (e.g., a belief or inference), to be the following position,

> **Supervenience Thesis**: The fact that E(s) supervenes solely on internal features of the agent.[22]

Supervenience Thesis regarding the rationality of our belief updating processes and resultant beliefs reads as follows,

> **U & B Supervenience Thesis**: The facts that (i) an agent's belief updating process is rational and (ii) the agent's resultant belief is doxastically justified supervene solely on internal features of the agent.

Ralph Wedgwood (2017: 163), for example, advocates for U & B Supervenience Thesis, amongst other internalist positions, in the following:

> Indeed, intuitively, internalism seems to articulate a completely general feature of rationality. It is not just rational belief-states that have this feature: the same feature seems to hold of rational processes of belief revision; and it also seems to hold of rational mental events and mental states of other kinds. . . . Whenever we assess any process of reasoning or mental state or event as rational or irrational, we are assessing it on the basis of its relation to the mental events and states that are present in the thinker's mind . . .[23]

22. Epistemic externalism is, thus, the denial of Supervenience Thesis. This is not the only way of characterizing the externalist/internalist divide. See (Pryor 2001) for further discussion.

23. Not everyone takes internalism to be a "completely general feature of rationality." Declan Smithies (2015), for example, argues for an access internalist account of propositional justification while accepting an externalist account of doxastic justification. According to Smithies, doxastic justification is externalist in virtue of the fact that we don't have direct access to epistemic basing facts, that is, facts about what attitudes serve as the epistemic bases of our beliefs. Nothing I say is strictly incompatible with an internalism about propositional justification—the position that the features of an epistemic circumstance that make a proposition rational to believe are internal

Wedgwood (2002; 2017), along with many other advocates of internalism (e.g., Langsam 2008; Pollock & Cruz 1999; Schafer 2014; Schoenfield 2015), motivate internalism on the grounds that mental states and processes mediate the effects of the extramental world on our personal-level belief updating processes. In using PA, our access to and operations over the abacus are mediated by various perceptual states. The obvious objection to my position is to argue that the belief updating processes involved in using PA extend only so far as one's perceptual access to the abacus, as opposed to being partially constituted by the states of and one's operations over the beads.

In §5.1 I respond to Wedgwood's argument for internalism, as Wedgwood's thoroughgoing internalism is clearly in conflict with Extramental Thesis. In §5.2, I develop a positive account of the epistemic functional role of perception in our use of extramental representations in updating our attitudes that is compatible with Extramental Thesis.

5.1. Internal States Mediate the Effects of the Extramental

Wedgwood begins his argument for internalism by claiming,

> [T]he facts about the attitudes and ways of thinking that are rational in your situation must be capable of directly guiding your thinking in this situation. . . . the fact that it is rational for you to form a belief—or some further fact that the fact in question supervenes on—must be capable of being the proximate explanation (at the folk-psychological level of explanation) of your actually forming that belief. (2017: 182)

In other words, Wedgwood claims that the facts of an epistemic circumstance, D, that make it the case that an agent is justified in adopting a belief, B(p)—that is, the facts that make p propositionally rational to believe in D—must be capable of directly guiding the agent's attitude updating processes. According to Wedgwood, a fact, F, directly guides an agent's adoption of B(p), in the relevant sense, iff F is the proximal cause of the agent's adopting B(p) according to a fully-articulated (and true) folk psychological explanation. Extramental facts can't directly guide an agent's belief updating processes in Wedgewood's sense; the effects of extramental facts will be mediated by proximal mental states (e.g., mental representations of those facts). Therefore, the facts that make p propositionally rational to believe must be internal.

features. Nonetheless, Smithies would likely object to an externalism about the rationality of our belief updating processes.

One can consistently accept Extramental Thesis and the claim that the features of an epistemic circumstance that make it propositionally justified to believe a proposition must, in principle, be capable of directly guiding an agent's attitude formation processes, and, thereby, must be internal. My position is compatible with an epistemic internalism about propositional justification. Although I am dubious of various aspects of Wedgwood's position (e.g., his focus on folk psychological explanation), Wedgwood has yet to make a claim to which I am forced to object. I grant, for the sake of argument, Wedgwood's internalism about propositional justification.

However, Wedgwood proceeds to argue,

> Intuitively, one might form a belief that it is rational for one to form through sheer dumb luck. . . . **In this case, you form a certain belief in a situation that stands in the rationalizing relation to forming that belief; but you are not in any way guided by the fact that your situation rationalizes your forming that belief**. . . . So, I propose, for you to be guided by the fact that you are in a situation in which it is rational for you to form a belief in this proposition, it must be no accident that you form this belief in a situation in which it is rational for you to do so. (2017: 182, emphasis mine)

Wedgwood argues that rationally updating one's attitudes—that is, updating in a manner that isn't sheer dumb luck—in adopting a doxastically justified belief, B(p), in an epistemic circumstance, D, requires that the features of D that make p propositionally justified to believe directly guide one's adoption of B(p). In other words, Wedgwood assumes the following,

An agent has rationally updated her attitudes in adopting B(p) in D only if

> **M**: the personal-level mental states that are the proximal causes (according to a fully-articulated folk psychological explanation) of the agent's adoption of B(p) in D

are identical to

> **F**: the features of D that make p propositionally rational to believe.

So, if (i) an agent has rationally updated her attitudes in adopting a belief, B(p), only if M and F are identical and (ii) F must be internal features of an agent, then U & B Supervenience Thesis is true; the (doxastic) rationality of one's updating processes and resultant beliefs will supervene solely on internal features of the agent.

Crucially, Wedgwood's argument rests on the claim that M and F are identical when one has rationally updated one's attitudes; however, this claim is false, and, therefore, Wedgwood's argument is unsound. We frequently update our attitudes in a manner that involves mental states at the personal level that serve as the proximal causes (according to a fully-articulated folk psychological explanation) of our adoption of a doxastically justified belief, B(p), and yet these mental states have nothing to do with why p is propositionally justified to believe in our epistemic circumstances. In other words, it's frequently the case that M and F are not identical, despite the fact that we have updated our attitudes rationally and adopted a doxastically justified belief.

As an example of a type of rational belief updating process in which M and F come apart, take counting algorithms. For example, Counting Min is a counting algorithm that involves determining the sum of two natural numbers by starting from the larger number (if one number is larger than the other) and counting on by a number of units equal to the smaller. Prior to learning addition tables, children frequently use counting algorithms to determine the sums of natural numbers (Fuson 2020; Lemaire, Barrett, Fayol, & Abdi 1994). Given the limitations on our innate cognitive resources for representing numbers (as discussed in §3.2), counting will involve tokening the names of successive numbers in a numeric code. So, imagine a child, Gustave, comes to believe that 7 + 25 = 32 by internally executing Counting Min using inner speech and the English language. The basic transitions in thought that Gustave performs at the personal level and that constitute his reasoning are transitions between a series of inner speech utterances. Given that inner speech constitutes (in part) an offline, sensory simulation of speaking that we generate by internally using our natural language competence (as discussed in §3.1), the causal antecedents of Gustave's inner speech utterances will involve (amongst other states) Gustave's,

1. Background beliefs regarding what certain number words sound like and how these words are syntactically constructed. (Gustave may not have committed to rote memory the English word "thirty-two"; nonetheless, Gustave may know the syntactic rules in English regarding how number words are formed so that he can generate "thirty-two" from the words he has committed to rote memory, i.e., "thirty" and "two" [Skwarchuk & Anglin 2002]),
2. Procedural beliefs about Counting Min, and
3. Mnemonic states regarding what numbers he is adding, how many units he has counted thus far, and, therefore, how many more units he still needs to count.

If Gustave aptly uses Counting Min through tokening a series of inner speech utterances, he has reasoned well in his adoption of the belief that 7 + 25 = 32. However, what makes it the case that Gustave is propositionally justified to believe that 7 + 25 = 32 has nothing to do with 1. In other words, there is no rationalizing relation between his resultant belief that 7 + 25 = 32 and his background beliefs about how certain number words sound in English. What makes it propositionally rational for Gustave to believe that 7 + 25 = 32 is (the content of) his background knowledge of natural numbers and addition; however, his background knowledge doesn't serve as the proximal cause of his conclusion.

What exactly does Gustave's background knowledge of natural numbers and addition consist in? Although a full discussion of this issue would be a paper unto itself, as there is an extensive literature on our understanding of the natural numbers, most theorists discuss background knowledge of the natural numbers in terms of (i) exact numerical equality, as defined by Hume's principle, and (ii) the successor function (Izard, Pica, Spelke, & Dehaene 2008; Leslie, Gelman, & Gallistel 2008; Schneider, Sullivan, Guo, & Barner 2021). Hume's principle—that two sets are equinumerous if and only if their elements can be placed in one-to-one correspondence—is central to set-theoretic definitions of the natural numbers provided by Frege and Russell. The successor function, on the other hand, is central to the Peano–Dedekind axioms for the natural numbers.

Theorists typically discuss our understandings of (i) exact numerical equality and (ii) the successor function as separate aspects of our knowledge of the natural numbers because there is good evidence that our understandings of i and ii develop independently and may undergird different mathematical abilities. Important for our purposes, there is good evidence that an understanding of the successor function is crucial to acquiring an understanding of addition and other arithmetic operations (Schneider et al. 2021). So, Gustave's background knowledge that provides propositional justification for his conclusion is his knowledge (under some guise or other) that addition is the repeated application of the successor function and, thus, that 7 + 25 is equivalent to the seventh successor of 25. However, when it comes to Gustave's use of this background knowledge through Counting Min, the information that proximally causes his execution of the algorithm—which consists in his tokening of a series of inner speech utterances of English number words—includes, for example, phonological information about English, which has nothing to do with Gustave's propositional justification for his conclusion.

In Gustave's case, M and F clearly come apart, and yet Gustave reasons impeccably by using Counting Min through a series of inner speech utterances. Gustave isn't lucky to arrive at a rational belief. Gustave's reasoning is a manifestation of his arithmetic competence, despite the fact that his reasoning isn't directly guided (in Wedgwood's sense of the term) by the features of his epistemic circumstances that make him propositionally justified to believe that 7 + 25 = 32.

5.2. The Epistemic Role of Perception in the Use of Extramental Representation

Now imagine that Gustave counts with a series of manual signs in American Sign Language (ASL). Just as using PA to perform a calculation doesn't involve concurrently and internally performing the same calculation in MA, overtly counting with a numeric code doesn't involve concurrently and internally performing the same counting sequence with that numeric code. Gustave counts on from 25 by 7 units until he reaches 32 through tokening a series of manual signs in ASL. In order for Gustave to update his attitudes rationally through the use of Counting Min, he actually has to count and, thus, repeatedly apply the successor function. But now Gustave's counting is realized in a series of manual signs (as opposed to inner speech utterances), and, thereby, whether Gustave has rationally updated his attitudes will directly depend on the manual signs he tokens and their semantic content in ASL. Similarly, Gustave isn't lucky to arrive at a rational belief in executing Counting Min with his hands. Gustave's belief updating process, which includes a series of extramental manual signs in ASL, is a manifestation of his competence with Counting Min.

In using manual signs, Gustave will experience various somatic and visual states regarding the positions and movements of his hands that are crucial in guiding his use of Counting Min. According to Wedgwood, if Gustave has updated his attitudes rationally (which he clearly has), these perceptual states—which directly guide aspects of the mental portion of Gustave's belief updating process—must be part of what makes it the case that Gustave is propositionally justified to believe $7 + 25 = 32$. But, again, clearly M and F come apart. Gustave's somatic and visual states regarding the positions and movements of his hands have nothing to do with the features of his epistemic circumstances that make it the case that $7 + 25 = 32$ is propositionally justified to believe.

An internalist might grant that M and F can come apart in cases of rational belief updating and still claim that when Gustave counts with manual signs in ASL he uses (the contents of) perceptual states regarding his hands as reason for his interim and final mathematical conclusions. Gustave's access to his manual signs is mediated by perception, and, thereby, perception functions as a source of evidence that Gustave uses in updating his attitudes. Gustave's belief updating process begins from his evidential base—which includes his perceptions and relevant background mathematical attitudes—and proceeds purely internally. Gustave's manual signs aren't properly part of his attitude updating process, nor are they directly relevant for rational evaluation (or so the argument goes).

Of course, I lack the space to respond to all plausible means of motivating epistemic internalism. Instead of providing piecemeal responses to various internalist arguments, I will close by offering a positive account of the role of perception in our belief updating processes that is consistent with Extramental Thesis. In the following, I argue that, in using something like an abacus system and PA or ASL and Counting Min, we do not use perceptual states (or the contents thereof) as reasons for our interim or final mathematical conclusions. Instead, perception plays a purely meta-reasoning role in guiding our use of a calculation algorithm.

Although much of the work in cognitive psychology on metacognition focuses on mnemonic processes, more recently Ackerman and Thompson have generated a model of meta-reasoning, or the metacognitive monitoring and control processes involved in reasoning (Ackerman & Thompson 2015; 2017a; 2017b). Metacognitive monitoring and control processes are important executive functions that afford us flexibility in regulating our thoughts. Meta-reasoning monitoring states are used in meta-level assessments of our first-order reasoning processes and in the allocation of attentional and working memory resources throughout reasoning. For instance, various meta-reasoning cues can result in a 'feeling of rightness' that we use to guide (i) our initial choice of decision procedure in reasoning, (ii) the allocation of cognitive effort to various tasks, and (iii) our decision to terminate reasoning and accept our conclusion, as opposed to rechecking our inferential steps or abandoning the reasoning process (Thompson & Morsanyi 2012; Thompson et al. 2013). Meta-reasoning monitoring states, like the feeling of rightness, serve as a crucial source of information about our reasoning itself, as opposed to the contents about which we are reasoning.

To see the importance of meta-reasoning in mental calculation—again, mental calculation being a process that loads on working memory and requires attentional resources to guide the use of a calculation algorithm—let's examine an actual case report of a patient (M.M.) with brain damage that selectively affected meta-reasoning monitoring and control procedures (Semenza, Miceli, & Girelli 1997). M.M.'s background knowledge of arithmetical facts and calculation procedures seemed to be intact. Nonetheless, M.M. frequently errored in multi-digit calculation, where the likelihood of error increased with the number of suboperations required to use the relevant calculation algorithm. M.M.'s errors were non-systematic; for instance, on different occasions, he incorrectly performed each suboperation of LM (e.g., carrying, determining the factors for calculating the partial products of the operands, recalling the relevant arithmetic facts, etc.) while exhibiting competence with each suboperation on other occasions. M.M. couldn't tell whether he was accurately preforming each suboperation and frequently asked the experimenters if he had finished calculating or whether additional suboperations needed to be performed. M.M's errors were not due to a systematic misunderstanding of calculation procedures or damaged background mathematical knowledge but an inability to monitor and control his use of this background knowledge in calculation.

As M.M. demonstrates, meta-reasoning plays a crucial role in guiding first-order reasoning; however, meta-reasoning monitoring states do not function as reasons for our conclusions. To claim that meta-reasoning states are just more reasons would set off a familiar Carrollinian (1895) regress. If our meta-reasoning monitoring states (MS) along with our first-order beliefs (FB) regarding, for example, the partial products of our operands, both function as reasons for our conclusions such that the inferential transitions we actually perform are from MS and FB (collectively) to our conclusions, then—because these larger inferences from MS and FB would also require working memory resources to guide their execution—our ability to reliably perform these inferential transitions would themselves require meta-reasoning monitoring states and control processes.

With regards to perception, the claim that perception can play monitoring and guiding roles is widely accepted in the contemporary literature on motor control. For example, in seeing a colleague, we may formulate the intention to talk about how the semester is developing. In speaking to our colleague, we will use perceptual feedback from our speech apparatus and from listening to ourselves speak to monitor and control our speech (Pickering & Garrod 2013). If we perceive ourselves uttering phonemes, morphemes, words, expressions, etc. that differ from what we intend, we will intervene and correct the error.

Analogously, I argue that when a person uses extramental representations in updating her beliefs with, say, PA or finger reckoning, she employs perceptual feedback from her muscles and tendons and visual perceptions of her hands to guide (at a meta-level) her operations over her extramental representations. Perceptual states (i) provide us access to how well we are performing extramental state transitions, like moving the beads of an abacus in summing the partial products of our operands, and (ii) allow us to intervene if we notice any errors in our execution. When we use perception to guide our extramental state transitions in using PA, finger reckoning, etc., perception functions at a meta-reasoning level. In other words, perceptions do not function as reasons for our conclusion but as monitoring states—crucial sources of information about our belief updating processes used in guiding our extramental operations.

If we were to insist that, say, Kaori uses information about her abacus and its bead structure, garnered from perception, as reason for her mathematical conclusion, then (to be consistent) we would also need to insist that Tetsuo uses information about the visuospatially imaged bead structure (garnered from his imagery) as reason for his conclusion as well. In fact, we'd have to insist that all of our arithmetic attitudes are, in part, epistemically based in information about the appearance of the numeric codes we employ in updating our attitudes, for example, information about the phonology of the language we use in inner speech to perform a mental calculation. As I've argued, given limits on our innate cognitive resources for representing exact numbers and quantities, mental calculation must involve the internal use of a numeric code. We do not possess exact number concepts in something like the Language of Thought that allow us to directly think about numbers in a manner unmediated by our internal use of a natural language, abacus system, etc. Information about, for example, the visuospatially imaged bead structure in using MA, like perceptual information about the structure of an actual abacus in using PA, operates at a meta-reasoning level; it provides us access to how our belief updating process is unfolding as opposed to functioning as reason for our mathematical conclusions. In using MA, Tetsuo does not employ information about the visuospatially imaged bead structure (e.g., the relative positions of the beads)—as opposed to the number denoted by that structure in the abacus system—as reason to adopt a particular mathematical conclusion. Tetsuo would not (nor should he!) cite as a reason for his conclusion that his visuospatially imaged abacus is represented as having beads in a certain configuration. Similarly, Gustave would not (nor should he!) cite as a reason for his conclusion information about what English number words sound like or what manual signs in ASL look like.[24] Gustave bases his belief in the numeric content of the English and ASL expressions he tokens, as opposed to information about how those expressions appear in various sensory modalities.

In sum, I've provided an account of how perception operates in PA that is consistent with Extramental Thesis. When using PA (and other belief updating processes that extend beyond the bounds of the mental) perceptual states function as meta-reasoning states used in monitoring and controlling our first-order belief updating processes. The fact that our access to and operations over extramental representations are mediated by various perceptual states is not a reason in itself to reject Extramental Thesis.

6. Conclusion

Traditionally, reasoning is assumed to be of central epistemic importance, insofar as reasoning is taken to be our personal-level means of establishing the reasons for which we possess the beliefs we do. However, as I've demonstrated, there is no good reason to restrict epistemic discussion to propositional attitudes and inferential transitions. As examples like PA and finger reckoning demonstrate, extramental representations and operations ought to be seen as part of our belief updating processes and as directly relevant to rational evaluation. We perform operations over extramental representations in updating our attitudes

24. Additionally, many theorists working on inner speech, despite disagreeing about various aspects of the phenomenon, accept that inner speech plays an important meta-reasoning role in governing our first-order reasoning processes (e.g., Fletcher & Carruthers 2012; Geurts 2018; Jackendoff 1996).

that exhibit our epistemic competences and are properly part of the process of determining our reasons for belief. As epistemologists, we need to shift our focus to a broader notion of a belief updating process that isn't purely mental. Reasoning, qua mental, rule-governed operation over propositional attitudes, isn't of central epistemic importance. It's belief updating processes that are of central epistemic importance, and these processes aren't solely mental.

Acknowledgements

I would like to thank two anonymous referees for their invaluable commentary. I would also like to thank Brian Weatherson for discussion and feedback on earlier drafts.

References

Ackerman, Rakefet and Valerie A. Thompson (2015). Meta-Reasoning. In Aidan Feeney and Valerie A. Thompson (Eds.), *Reasoning as Memory* (164–82). Psychology Press.

Ackerman, Rakefet and Valerie A. Thompson (2017a). Meta-Reasoning: Monitoring and Control of Thinking and Reasoning. *Trends in Cognitive Sciences, 21*(8), 607–17.

Ackerman, Rakefet and Valerie A. Thompson (2017b). Meta-Reasoning: Shedding Meta-Cognitive Light on Reasoning Research. In Linden J. Ball and Valerie A. Thompson (Eds.), *The Routledge International Handbook of Thinking and Reasoning* (1–15). Psychology Press.

Alderson-Day, Ben and Charles Fernyhough (2015). Inner Speech: Development, Cognitive Functions, Phenomenology, and Neurobiology. *Psychological Bulletin, 141*(5), 931–65.

Ardila, Alfredo and Mónica Rosselli (2002). Acalculia and Dyscalculia. *Neuropsychology Review, 12*(4), 179–231.

Audi, Robert (1986). Belief, Reason, and Inference. *Philosophical Topics, 14*(1), 27–65.

Baddeley, Alan (2007). *Working Memory, Thought, and Action*. Oxford University Press.

Baddeley, Alan (2010). Working Memory. *Current Biology, 20*(4), R136–R140.

Baddeley, Alan (2017). *Exploring Working Memory: Selected Works of Alan Baddeley*. Routledge.

Barner, David, George Alvarez, Jessica Sullivan, Neon Brooks, Mahesh Srinivasan, and Michael C. Frank (2016). Learning Mathematics in a Visuospatial Format: A Randomized, Controlled Trial of Mental Abacus Instruction. *Child Development, 87*(4), 1146–58.

Benson, D. Frank and Martha B. Denckla (1969). Verbal Paraphasia as a Source of Calculation Distrubance. *Archives of Neurology, 21*(1), 96–102.

Beran, Michael J. (2001). Summation and Numerousness Judgments of Sequentially Presented Sets of Items by Chimpanzees (*Pan troglodytes*). *Journal of Comparative Psychology, 115*(2), 181–91.

Bergmann, Michael (2006). *Justification without Awareness: A Defense of Epistemic Externalism*. Oxford University Press.
Boghossian, Paul (2014). What Is Inference? *Philosophical Studies, 169*(1), 1–18.
Boghossian, Paul (2018). Delimiting the Boundaries of Inference. *Philosophical Issues, 28*(1), 55–69.
Boghossian, Paul (2019). Inference, Agency and Responsibility. In Magdalena Balcerak Jackson and Brendan Balcerak Jackson (Eds.), *Reasoning: New Essays on Theoretical and Practical Thinking* (101–24). Oxford University Press.
Brannon, Elizabeth M., Kerry E. Jordan, and Sarah M. Jones (2010). Behavioral Signatures of Numerical Cognition. In Michael Platt and Asif Ghazanfar (Eds.), *Primate Neuroethology* (144–59). Oxford University Press.
Broncano-Berrocal, Fernando (2014). Is Safety in Danger? *Philosophia, 42*(1), 1–19.
Brooks, Neon B., David Barner, Michael Frank, and Susan Goldin-Meadow (2018). The Role of Gesture in Supporting Mental Representations: The Case of Mental Abacus Arithmetic. *Cognitive Science, 42*(2), 554–75.
Broome, John (2013). *Rationality through Reasoning*. Wiley-Blackwell.
Broome, John (2014). Normativity in Reasoning. *Pacific Philosophical Quarterly, 95*(4), 622–33.
Brown, D. G. (1955). The Nature of Inference. *The Philosophical Review, 64*(3), 351–69.
Burge, Tyler (1993). Content Preservation. *Philosophical Review, 102*(4), 457–88.
Cajori, Florian (1993). *A History of Mathematical Notations*. Dover.
Carroll, Lewis (1895). What the Tortoise Said to Achilles. *Mind, 4*(14), 278–280.
Conee, Earl and Richard Feldman (2004). *Evidentialism: Essays in Epistemology*. Clarendon Press.
Conee, Earl and Richard Feldman (2008). Evidence. In Quentin Smith (Ed.), *Epistemology: New Essays* (83–104). Oxford University Press.
Dancy, Jonathan (2000). *Practical Reality*. Oxford University Press.
De Guerrero, Maria C. M. (2005). *Inner Speech—L2: Thinking Words in a Second Language*. Springer.
Delazer, Margarete and Lisa Bartha (2001). Transcoding and Calculation in Aphasia. *Aphasiology, 15*(7), 649–79.
Delazer, Margarete, Luisa Girelli, Carlo Semenza, and Gianfranco Denes (1999). Numerical Skills and Aphasia. *Journal of the International Neuropsychological Society, 5*(3), 213–21.
Dougherty, Trent (2011). In Defense of Propositionalism About Evidence. In Trent Dougherty (Ed.), *Evidentialism and Its Discontents* (226–33). Oxford University Press.
Feigenson, Lisa, Stanislas Dehaene, and Elizabeth Spelke (2004). Core Systems of Number. *Trends in Cognitive Sciences, 8*(7), 307–14.
Flavell, John H., Frances L. Green, Eleanor R. Flavell, and James B. Grossman (1997). The Development of Children's Knowledge About Inner Speech. *Child Development, 68*(1), 39–47.
Fletcher, Logan and Peter Carruthers (2012). Metacognition and Reasoning. *Philosophical Transactions of the Royal Society B: Biological Sciences, 367*(1594), 1366–78.
Frank, Michael C. and David Barner (2012). Representing Exact Number Visually Using Mental Abacus. *Journal of Experimental Psychology: General, 141*(1), 134–39.

Frank, Michael C., Daniel L. Everett, Evelina Fedorenko, and Edward Gibson (2008). Number as a Cognitive Technology: Evidence from Pirahã Language and Cognition. *Cognition, 108*(3), 819–24.

Frank, Michael C., Evelina Fedorenko, and Edward Gibson (2008). *Language as a Cognitive Technology: English-Speakers Match Like Pirahã When You Don't Let Them Count.* Paper presented at the Proceedings of the Annual Meeting of the Cognitive Science Society.

Fuson, Karen C. (2020). An Analysis of the Counting-on Solution Procedure in Addition. In Thomas P. Carpenter, James M. Moser, and Thomas A. Romberg (Eds.), *Addition and Subtraction* (67–81). Routledge.

Gauker, Christopher (2018). Inner Speech as the Internalization of Outer Speech. In Peter Langland-Hassan and Agustin Vicente (Eds.), *Inner Speech: New Voices* (53–77). Oxford University Press.

Geurts, Bart (2018). Making Sense of Self Talk. *Review of Philosophy and Psychology, 9*(2), 271–85.

Geva, Sharon and Charles Fernyhough (2019). A Penny for Your Thoughts: Children's Inner Speech and Its Neuro-Development. *Frontiers in Psychology, 10*, 1–12.

Glüer, Kathrin and Åsa Wikforss (2009). Against Content Normativity. *Mind, 118*(469), 31–70.

Glüer, Kathrin and Åsa Wikforss (2013). Against Belief Normativity. In Timothy Chan (Ed.), *The Aim of Belief* (80–99). Oxford University Press.

Goldman, Alvin I. (1979). What Is Justified Belief? In Ernest Sosa and Jaegwon Kim (Eds.), *Epistemology: An Anthology* (340–53). Blackwell.

Goldman, Alvin I. (1986). *Epistemology and Cognition*. Harvard University Press.

Gordon, Peter (2004). Numerical Cognition without Words: Evidence from Amazonia. *Science, 306*(5695), 496–99.

Grandchamp, Romain, Lucile Rapin, Marcela Perrone-Bertolotti, Cédric Pichat, Célise Haldin, Emilie Cousin, . . . Hélène Lœvenbruck (2019). The Condialint Model: Condensation, Dialogality, and Intentionality Dimensions of Inner Speech within a Hierarchical Predictive Control Framework. *Frontiers in Psychology, 10*, 1–30. https://doi.org/10.3389/fpsyg.2019.02019

Greco, John (1999). Agent Reliabilism. *Philosophical Perspectives, 13*, 273–96.

Greco, John (2003). Knowledge as Credit for True Belief. In Michael DePaul and Linda Zagzebski (Eds.), *Intellectual Virtue: Perspectives from Ethics and Epistemology* (111–34). Oxford University Press.

Gregory, Daniel (2016). Inner Speech, Imagined Speech, and Auditory Verbal Hallucinations. *Review of Philosophy and Psychology, 7*(3), 653–73.

Hatano, Giyoo, Yoshio Miyake, and Martin G. Binks (1977). Performance of Expert Abacus Operators. *Cognition, 5*(1), 47–55.

Heavey, Christopher L. and Russell T. Hurlburt (2008). The Phenomena of Inner Experience. *Consciousness and Cognition, 17*(3), 798–810.

Hishitani, Shinsuke (1990). Imagery Experts: How Do Expert Abacus Operators Process Imagery? *Applied Cognitive Psychology, 4*(1), 33–46.

Izard, Véronique, Pierre Pica, Elizabeth S. Spelke, and Stanislas Dehaene (2008). Exact Equality and Successor Function: Two Key Concepts on the Path Towards Understanding Exact Numbers. *Philosophical Psychology, 21*(4), 491–505.

Jackendoff, Ray (1996). How Language Helps Us Think. *Pragmatics & Cognition, 4*(1), 1–34.

Knifong, J. Dan and Grace M. Burton (1979). Chisanbop: Just Another Kind of Finger Reckoning? *The Arithmetic Teacher, 26*(7), 14–17.

Korcz, Keith Allen (1997). Recent Work on the Basing Relation. *American Philosophical Quarterly, 34*(2), 171–91.

Korcz, Keith Allen (2000). The Causal-Doxastic Theory of the Basing Relation. *Canadian Journal of Philosophy, 30*(4), 525–50.

Koziolek, Nicholas (2021). Inferring as a Way of Knowing. *Synthese, 198*(Suppl 7), 1563–82.

Langsam, Harold (2008). Rationality, Justification, and the Internalism/Externalism Debate. *Erkenntnis, 68*(1), 79–101.

Lee, Kyoung-Min and So-Young Kang (2002). Arithmetic Operation and Working Memory: Differential Suppression in Dual Tasks. *Cognition, 83*(3), B63–B68.

LeFevre, Jo-Anne, Diana DeStefano, Benjamin Coleman, and Tina Shanahan (2005). Mathematical Cognition and Working Memory. In J. I. D. Campbell (Ed.), *Handbook of Mathematical Cognition* (361–77). Psychology Press.

Lemaire, Patrick, Susan E. Barrett, Michael Fayol, and Hervé Abdi (1994). Automatic Activation of Addition and Multiplication Facts in Elementary School Children. *Journal of Experimental Child Psychology, 57*(2), 224–58.

Leslie, Alan M., Rochel Gelman, and C. R. Gallistel (2008). The Generative Basis of Natural Number Concepts. *Trends in Cognitive Sciences, 12*(6), 213–18.

Lidstone, Jane S. M., Elizabeth Meins, and Charles Fernyhough (2010). The Roles of Private Speech and Inner Speech in Planning During Middle Childhood: Evidence from a Dual Task Paradigm. *Journal of Experimental Child Psychology, 107*(4), 438–51.

Miller, J. T. M. (2020). The Ontology of Words: Realism, Nominalism, and Eliminativism. *Philosophy Compass, 15*(7), 1–13. https://doi.org/10.1111/phc3.12691

Mitova, Veli (2015). Truthy Psychologism About Evidence. *Philosophical Studies, 172*(4), 1105–26.

Munroe, Wade (2022). Why Are You Talking to Yourself? The Epistemic Role of Inner Speech in Reasoning. *Noûs, 56*(4), 841–66.

Munroe, Wade (2023a). Thinking through Talking to Yourself: Inner Speech as a Vehicle of Conscious Reasoning. *Philosophical Psychology, 36*(2), 292–318.

Munroe, Wade (2023b). Semiotics in the Head: Thinking About and Thinking Through Symbols. *Philosophy and Phenomenological Research, 107*(2), 413–438. https://doi.org/10.1111/phpr.12923

Nagel, Jennifer (2013). Knowledge as a Mental State. In Tamar Szabó Gendler and John Hawthorne (Eds.), *Oxford Studies in Epistemology* (Vol. 4, 275–310). Oxford University Press.

Neta, Ram (2008). What Evidence Do You Have? *The British Journal for the Philosophy of Science, 59*(1), 89–119.

Neta, Ram (2019). The Basing Relation. *Philosophical Review, 128*(2), 179–217.

O'Shaughnessy, David M., Edward Gibson, and Steven T. Piantadosi (2021). The Cultural Origins of Symbolic Number. *Psychological Review, 129*(6), 1442–56.

Perrone-Bertolotti, Marcela, Lucile Rapin, J.-P. Lachaux, Monica Baciu, and Héléne Lœvenbruck (2014). What Is That Little Voice inside My Head? Inner Speech Phe-

nomenology, Its Role in Cognitive Performance, and Its Relation to Self-Monitoring. *Behavioural Brain Research, 261*, 220–39.
Pica, Pierre, Cathy Lemer, Véronique Izard, and Stanislas Dehaene (2004). Exact and Approximate Arithmetic in an Amazonian Indigene Group. *Science, 306*(5695), 499–503.
Pickering, Martin J. and Simon Garrod (2013). An Integrated Theory of Language Production and Comprehension. *Behavioral and Brain Sciences, 36*(4), 329–47.
Pollock, John L. and Joseph Cruz (1999). *Contemporary Theories of Knowledge* (2nd ed., Vol. 35). Rowman & Littlefield.
Pritchard, Duncan (2008). Sensitivity, Safety, and Anti-Luck Epistemology. In John Greco (Ed.), *The Oxford Handbook of Skepticism* (437–55). Oxford University Press.
Pryor, James (2001). Highlights of Recent Epistemology. *The British Journal for the Philosophy of Science, 52*(1), 95–124.
Quilty-Dunn, Jake and Eric Mandelbaum (2018). Inferential Transitions. *Australasian Journal of Philosophy, 96*(3), 532–47.
Quilty-Dunn, Jake and Eric Mandelbaum (2019). Non-Inferential Transitions: Imagery and Association. In Timothy Chan and Anders Nes (Eds.), *Inference and Consciousness* (151–71). Routledge.
Raghubar, Kimberly P., Marcia A. Barnes, and Steven A. Hecht (2010). Working Memory and Mathematics: A Review of Developmental, Individual Difference, and Cognitive Approaches. *Learning and Individual Differences, 20*(2), 110–22.
Ryle, Gilbert (1949). *The Concept of Mind*. Hutchinson's University Library.
Schafer, Karl (2014). Doxastic Planning and Epistemic Internalism. *Synthese, 191*(12), 2571–91.
Schneider, Rose M., Jessica Sullivan, Kaiqi Guo, and David Barner (2021). What Counts? Sources of Knowledge in Children's Acquisition of the Successor Function. *Child Development, 92*(4), e476–e492.
Schoenfield, Miriam (2015). Internalism without Luminosity. *Philosophical Issues, 25*(1), 252–72.
Semenza, Carlo, Laura Miceli, and Luisa Girelli (1997). A Deficit for Arithmetical Procedures: Lack of Knowledge or Lack of Monitoring? *Cortex, 33*(3), 483–98.
Shanon, Benny (1984). The Polyglot Mismatch and the Monolingual Tie: Observations Regarding the Meaning of Numbers and the Meaning of Words. *New Ideas in Psychology*, *2*(1), 75–79.
Silva, Paul and Luis R. G. Oliveira (2020). Propositional Justification and Doxastic Justification.
Skwarchuk, Sheri-Lynn and Jeremy M. Anglin (2002). Children's Acquisition of the English Cardinal Number Words: A Special Case of Vocabulary Development. *Journal of Educational Psychology, 94*(1), 107–25.
Smithies, Declan (2015). Why Justification Matters. In David Henderson and John Greco (Eds.), *Epistemic Evaluation: Point and Purpose in Epistemology* (224–44). Oxford University Press.
Sosa, Ernest (2010). How Competence Matters in Epistemology. *Philosophical Perspectives, 24*, 465–75.
Sosa, Ernest (2015). *Judgment & Agency*. Oxford University Press.
Spaepen, Elizabet, Marie Coppola, Elizabeth S. Spelke, Susan E. Carey, and Susan Goldin-Meadow (2011). Number without a Language Model. *Proceedings of the National Academy of Sciences, 108*(8), 3163–68.

Stigler, James W. (1984). "Mental Abacus": The Effect of Abacus Training on Chinese Children's Mental Calculation. *Cognitive Psychology, 16*(2), 145–76.
Sylvan, Kurt (2016). Epistemic Reasons II: Basing. *Philosophy Compass, 11*(7), 377–89.
Sylvan, Kurt and Errol Lord (in press). Prime Time (for the Basing Relation). In Patrick Bondy and J. Adam Carter (Eds.), *Well-Founded Belief: New Essays on the Basing Relation*. Routledge.
Tanaka, Satoshi, Keiko Seki, Takashi Hanakawa, Madoka Harada, Sho K. Sugawara, Norihiro Sadato, . . . Manabu Honda (2012). Abacus in the Brain: A Longitudinal Functional MRI Study of a Skilled Abacus User with a Right Hemispheric Lesion. *Frontiers in Psychology, 3*, 1–13.
Thompson, Valerie A. and Kinga Morsanyi (2012). Analytic Thinking: Do You Feel Like It? *Mind & Society, 11*(1), 93–105.
Thompson, Valerie A., Jamie A. Prowse Turner, Gordon Pennycook, Linden J. Ball, Hannah Brack, Yael Ophir, and Rakefet Ackerman (2013). The Role of Answer Fluency and Perceptual Fluency as Metacognitive Cues for Initiating Analytic Thinking. *Cognition, 128*(2), 237–51.
Tian, Xing and David Poeppel (2012). Mental Imagery of Speech: Linking Motor and Perceptual Systems through Internal Simulation and Estimation. *Frontiers in Human Neuroscience, 6*, 1–11.
Toppinen, Teemu (2015). How Norms (Might) Guide Belief. *International Journal of Philosophical Studies, 23*(3), 396–409.
Turri, John (2011). Believing for a Reason. *Erkenntnis, 74*(3), 383–97.
Valaris, Markos (2017). What Reasoning Might Be. *Synthese, 194*(6), 2007–24.
Van Rinsveld, Amandine, Laurence Dricot, Mathieu Guillaume, Bruno Rossion, and Christine Schiltz (2017). Mental Arithmetic in the Bilingual Brain: Language Matters. *Neuropsychologia, 101*, 17–29.
Warren, Jared (2022). Functionalism about Inference. *Inquiry*. Advance online publication. https://doi.org/10.1080/0020174X.2022.2075452
Wedgwood, Ralph (2002). Internalism Explained. *Philosophy and Phenomenological Research, 65*(2), 349–69.
Wedgwood, Ralph (2017). *The Value of Rationality*. Oxford University Press.
Welsh, Paul (1957). On the Nature of Inference. *Philosophical Review, 66*(4), 509–24.
Whitford, Thomas J., Bradley N. Jack, Daniel Pearson, Oren Griffiths, David Luque, Anthony W. F. Harris, . . . Mike E. Le Pelley (2017). Neurophysiological Evidence of Efference Copies to Inner Speech. *Elife, 6*, 1–26.
Williams, Burma P. and Richard S. Williams (1995). Finger Numbers in the Greco-Roman World and the Early Middle Ages. *Isis, 86*(4), 587–608.
Williamson, Timothy (2000). *Knowledge and Its Limits*. Oxford University Press
Wilson, Michael L., Marc D. Hauser, and Richard W. Wrangham (2001). Does Participation in Intergroup Conflict Depend on Numerical Assessment, Range Location, or Rank for Wild Chimpanzees? *Animal Behaviour, 61*(6), 1203–16.
Winters, Barbara (1980). Reasonable Believing. *Dialectica, 34*(1), 3–16.
Wolpert, Daniel M., Zoubin Ghahramani, and Michael I. Jordan (1995). An Internal Model for Sensorimotor Integration. *Science, 269*(5232), 1880–82.