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# Securing Arithmetical Determinacy

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The existence of non-standard models of first-order Peano-Arithmetic (PA) threatens to undermine the claim of the moderate mathematical realist that non-mysterious access to the natural number structure is possible on the basis of our best arithmetical theories. The move to logics stronger than FOL is denied to the moderate realist on the grounds that it merely shifts the indeterminacy "one level up" into the meta-theory by, illegitimately, assuming the determinacy of the notions needed to formulate such logics. This paper argues that the challenge can be met. We show how the quantifier "there are infinitely many" can be uniquely determined in a naturalistically acceptable fashion and thus be used in the formulation of a theory of arithmetic. We compare the approach pursued here with Field's justification of the same device and the popular strategy of invoking a second-order formalism, and argue that it is more robust than either of the alternative proposals.

## 1. Introduction

*Mathematical realists* hold that the statements of mathematics are determinately true or false and that they are so because of the mathematical objects, properties and relations referred to in these statements. Naturalistically minded mathematical realists, so called *moderate mathematical realists*, additionally demand that reference to the objects and structures responsible for the truth or falsity of mathematical statements be achieved through scientifically acceptable, non-mysterious means.

Moderate realists face a sceptical challenge: to explain how, given the abstract nature of mathematics, successful reference to mathematical structures and entities is possible without invoking scientifically unacceptable means. How, that is, reference of mathematical terms can be successfully fixed so as to afford determinacy of truth-value without invoking a scientifically unsupported "mathematical sense." The natural reply that reference is achieved through

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description of the relevant structures by our best mathematical theories is undermined by the fact that the usual languages used to formulate these theories are insufficient to guarantee determinate reference.

An attractive option for the moderate mathematical realist consists in moving to a richer framework in which it is possible to establish *categoricity theorems*. These results demonstrate that her theories uniquely fix the meanings of the mathematical terms and thus successfully achieve determinate reference. Unfortunately, this line of response appears to only exacerbate the original issue, for the referential determinacy of the resources required to achieve categorical characterizations of important mathematical theories, such as arithmetic, is itself in need of justification. Compounded by the fact that these resources are often much more complex than the notions whose reference they are used to fix, it is highly questionable whether the realist can legitimately appeal to them in her response to the sceptic without begging the question. This leaves the moderate mathematical realist in a vulnerable position.

The goal of this paper is to investigate the determinacy challenge against the moderate realist and argue that it can be met. It aims to show that the sceptical attack levelled against the resources appealed to by moderate mathematical realists to achieve a categorical characterization of arithmetic can be successfully answered through use of the generalized quantifier "there exist infinitely many."

The structure of the paper is as follows: §2 will outline the sceptical challenge for the moderate mathematical realist and sketch two well-known responses to it, one in terms of the generalized quantifier logic $\mathscr{L}(Q_0)$, the other in terms of second-order logic (SOL). §3 constitutes the core of this paper and presents my defense of the use of $\mathscr{L}(Q_0)$ in defusing the sceptical challenge. §4 compares this strategy with two existing approaches addressing the same issue: a defense of $\mathscr{L}(Q_{<\omega})$ by Field (1994; 2001) and a defense of SOL by Murzi and Topey (2021). §5 concludes. A short appendix contains the proofs of the results referred to in the main part of the paper.

## 2. The Skeptical Challenge for the Moderate Realist

The challenge for the realist derives from the fact that mathematical entities unlike, say, botanical entities, are *abstract* and thus inaccessible through any of the (accepted) senses. The moderate realist about arithmetic thus needs to supply a story of how successful reference to the natural number structure[1] can be

1. As is common, we identify mathematical structures with *isomorphism types*, i.e., classes of structures closed under isomorphism. The *natural number structure* is thus the class consisting of all structures isomorphic to $\mathbb{N}$. Slightly abusing terminology we also refer to this class itself as $\mathbb{N}$.

achieved without resorting to a scientifically unwarranted *mathematical sense*.[2] The natural reply that determinate reference is achieved through description, by means of our (best) mathematical theories of the structures we wish to talk about, is immediately met with a demand for the specification of the resources involved in formulating the respective theories. Here, the moderate realist avails herself of the apparatus of first-order logic (FOL) combined with a model-theoretic semantics, causing the skeptical challenge to gather pace.[3]

### *2.1. Non-standard Models of Arithmetic*

Achieving determinate reference to *the* natural-number structure amounts, in the context of model-theory, to the demand that our best theory of arithmetic be *categorical*, that it "pick out" a unique isomorphism-type.[4] If the framework in which such a theory is formulated is that of *first-order logic* (FOL) this demand cannot be met: the expressive resources of FOL are too limited to rule out *non-standard models of arithmetic*, i.e., models that satisfy our best first-order description of the natural number structure (say, for concreteness, *Peano-Arithmetic* (PA)) but that are not isomorphic to $\mathbb{N}$.

The inability of FOL to categorically characterize the natural-number structure runs deep: Button & Walsh (2018: 161) show that *no* compact logic[5] is capable of rendering a theory of arithmetic categorical, for in any such logic it is possible to add non-standard elements (e.g., infinitely large numbers) to the standard model of arithmetic to create an elementarily equivalent non-standard model. Compactness and its close relative, *completeness*, are highly priced properties of logical systems. The crucial question taken up below concerns whether they are necessary to possess a determinate grasp of a notion.

This, then, is the issue in a nutshell: a framework like that provided by FOL is "too weak" to uniquely pin down the standard model of arithmetic (up to isomorphism) and thus to ensure determinate reference of arithmetical terms.

2. This has been termed the *doxological* (Button and Walsh 2018) or *metasemantic* (Warren and Waxman 2020) challenge for the mathematical realist. See (Button and Walsh 2018) for an authoritative presentation which we are following closely.

3. Skeptical challenges regarding determinacy of reference are nothing new for the mathematical realist (see, e.g., (Benacerraf 1965)). They remain extremely pervasive and arise not just at the level of reference to individual mathematical objects ("the number 2"), but also at the level of entire mathematical structures ("*the* natural number structure"). It is this latter type of (in)determinacy that is at issue in the current paper and most of the debate at large.

4. See any introduction to model-theory for precise definition of this concept. A desirable consequence of categoricity is that categorical theories are *complete*, i.e., for any sentence $\varphi$ of the language of the theory $T$, if $T$ is categorical then either $T \vDash \varphi$ or $T \vDash \neg\varphi$. Hence, a categorical theory ensures determinacy of truth-value for any sentence of the language.

5. A logic is *compact* if, whenever $\Gamma \vDash \varphi$, there already exists a finite $\Gamma_0 \subseteq \Gamma$, s.t. $\Gamma_0 \vDash \varphi$.

FOL, however, is just an instrument used to formally codify antecedent beliefs of mathematicians about the properties of the mathematical structure $\mathbb{N}$. What the lack of categoricity of PA seems to show is that a first-order formalization of arithmetic is unable to properly capture *all* of the mathematician's antecedently held beliefs about that structure, that she believes or knows more about it than what can be adequately expressed in the confines of FOL.

There are two prominent diagnoses of the shortcoming exhibited in the admission of non-standard models and, correspondingly, two ways to address this perceived deficiency. Each motivates one of the enrichments of the formalism of FOL discussed in the next subsection. The first consists in maintaining that we know more about the subsets of $\mathbb{N}$ than FOL says we know. This analysis corresponds to a move to SOL with its quantification over subsets and a strengthening of the induction-schema of first-order PA. The second points out that the non-standard models of PA rely on a "misshapen" number sequence and claims that we have a firmer hold on the notion and properties of an $\omega$-sequence (i.e., a progression of elements mirroring the natural number sequence) than FOL admits. This diagnosis naturally corresponds to the use of a formalism—exemplified by $\mathscr{L}(Q_0)$ below,—that permits the moderate mathematical realist to adequately express more features of such a sequence than can be done in FOL.[6]

## 2.2. *Categorically Characterizing Arithmetic*

Given the limitations of FOL against the standard framework of model-theory how can the moderate realist claim to unambiguously refer to the natural number structure, the isomorphism-class of $\mathbb{N}$? There are several ways in which she might strengthen or modify FOL to achieve this goal. We will briefly survey the two approaches that we wish to contrast in the following: (i) second-order logic (SOL) and (ii) $\mathscr{L}(Q_0)$—FOL extended by the generalized quantifier "there exist (in)finitely many."[7]

(i) **Second-order Logic** (SOL)
The formalism of second-order logic is like the formalism of FOL, except that it also contains predicate variables $X, Y, Z$ in addition to individual variables $x, y, z$ and existential and universal quantifiers that bind these predicate-variables. Semantically, these quantifiers range over subsets/

6. There are many more ways to achieve categorical characterizations of arithmetic by strengthening the formal framework or imposing additional constraints on interpretations. For overview and discussion see (Button and Walsh 2016; 2018).

7. See (Shapiro 1991) for an introduction to second-order logic and (Barwise and Feferman 1985) for a study of generalized quantifier logics.

appropriate relations over the domain of a model, though there is space for variation: in a *full* semantics the quantifiers are taken to range over the full powerset of the appropriate Cartesian product of the domain. In a *Henkin-semantics* the quantifiers range over a subset thereof, subject to certain closure conditions (see (Shapiro 1991) or (Väänänen 2021a) for details). $PA_2$, Peano-Arithmetic in second-order logic, is just like PA, except that the induction-schema of PA can now be expressed by means of the single sentence:

$$\forall X[X(0) \wedge \forall x(X(x) \rightarrow X(s(x))) \rightarrow \forall x Xx]$$

The difference in expressive resources, coupled with the "right" semantics, matters, for it enables *Dedekind's Categoricity Theorem* (see (Dedekind 1888) and (Shapiro 1991)):

**Theorem:** Given full semantics, $PA_2$ is categorical.

With the increase in expressive power goes along a loss in (attractive) metalogical properties: SOL with full semantics is not compact and its consequence relation is not recursively axiomatizable. SOL with Henkin-semantics, on the other hand, can be interpreted as an essentially first-order formalism and thus inherits all the strengths and weaknesses of FOL itself.

(ii) The quantifiers **there are finitely many** and **there are infinitely many**
The concepts "finitely many" and "infinitely many" are not first-order definable. Adding a way to express either of these notions in FOL thus increases the expressive strength of the resulting system. This can be done through the device of a *Lindström-* or *generalized quantifier* (see (Peters and Westerståhl 2006)).
A (type $\langle 1 \rangle$) generalized quantifier is a (class-)function associating with every set a subset of its powerset. The quantifier "there exist finitely many," for example, is the function $\mathcal{Q}_{<\omega}$ such that, for every set $M$: $\mathcal{Q}_{<\omega}(M) = \{A \subseteq M \mid |A| < \omega\}$. The quantifier "there are infinitely many" is the function $\mathcal{Q}_0$, such that $\mathcal{Q}_0(M) = \{A \subseteq M \mid \omega \leq |A|\}$.
Adding a new type-appropriate quantifier-symbol $Q$ to the language of FOL, with formation-rules for formulas containing it analogous to those involving the existential and universal quantifier, satisfaction is defined as follows (where $\mathcal{M}$ is a model with domain $M$):

$$\mathcal{M} \vDash Qx\,\varphi(x) \text{ iff } \{a \in M \mid \mathcal{M} \vDash \varphi(a)\} \in \mathcal{Q}(M)$$

In FOL extended with either of the quantifiers $\mathcal{Q}_{<\omega}$ or $\mathcal{Q}_0$ we can now express, in the language of arithmetic, that every element has only finitely many predecessors: (F) $\forall x Q_{<\omega} y\, (y < x)$ or $\forall x \neg Q_0 y\, (y < x)$. This suffices to rule out non-standard models of arithmetic, thereby providing for the categoricity of PA + (F) in $\mathscr{L}(Q_{<\omega})$ or $\mathscr{L}(Q_0)$, respectively.

The expressive strength of $\mathscr{L}(Q_{<\omega})/\mathscr{L}(Q_0)$ falls somewhere between FOL and SOL. Compactness is lost and the consequence relation of $\mathscr{L}(Q_{<\omega})/\mathscr{L}(Q_0)$ is not recursively axiomatizable. However, the logics $\mathscr{L}(Q_{<\omega})$ and $\mathscr{L}(Q_0)$ are still first-order in that only individual variables are bound and quantifiers "merely" range over the first-order domain of the respective model.

### 2.3. *"Just more theory"*

Adopting a logic more expressive than FOL, such as SOL or $\mathscr{L}(Q_0)$, appears to resolve the indeterminacy of reference to the natural number structure. Why should the move to a more powerful formal framework be denied to the moderate realist? The worry associated with the use of a logic substantially stronger than FOL is that one is making use of resources one cannot claim to have a firmer hold on than one has on $\mathbb{N}$ itself. In other words, one might be accused of invoking notions reference to which is at least as problematic as reference to the natural number structure itself.

*Dedekind's Categoricity Theorem*, for example, requires that the semantics of SOL be *full*. However, in virtue of what can we be certain that the range of the second-order quantifiers is the complete powerset of the domain rather than a (suitably inclusive) subset thereof, that the semantics is *full*, rather than *Henkin*? After all, nothing about the formalism itself forces the quantifiers to take on the full interpretation, its use is perfectly consistent with possessing a Henkin semantics. What is needed to guarantee the *intended* full interpretation—to show that we have secured determinate reference of the second-order quantifiers—is yet *another* categoricity theorem, a proof that the use of the formalism determines a unique semantics.

But this means that the issue of determinately referring to the natural number structure has not been solved; it has merely been shifted to the level of the semantics of the logic used to characterize it. This is far from resolving the original issue of referential indeterminacy:[8]

> In trying to spell out why $PA_2$ picks out the [standard model of arithmetic], our moderate modelist appeals to Dedekind's Theorem … . In order

8. See, e.g., (Weston 1976) and (Putnam 1980) for this observation. See (Button and Walsh 2018: 158) for further references.

> for that Theorem to do the job she wants it to, she must have ruled out the Henkin semantics for second-order logic … . … And, so the worry goes, the distinction between full and Henkin models is *just more theory*, and hence up for reinterpretation. (Button and Walsh 2018: 159)

Similar worries apply in the case of $\mathscr{L}(Q_0)$: since $Q_0$ is indefinable in FOL its semantics involves concepts that go beyond FOL, and the question arises how determinate reference to *these* is secured. This, coupled with the possibility of non-standard interpretations of the notion of *infinity* in the background theory (see (Field 1994: 397) and (Parsons 2007: §48)) adds force to the challenge of the model-theoretic sceptic.[9]

The move to logics stronger than FOL to achieve categoricity therefore seems to only "shift[…] the problem from the identification of postulates characterizing ℕ categorically ('completely') into the semantics and model-theory of the logic used to state the postulates" (Read 1997: 91), and it is questionable whether any referential determinacy is gained through this. The "doxological challenge" confronting the moderate realist is lifted into the meta-theory—the bump in the carpet has merely been moved. How, then, in light of the possibility of non-standard interpretations of the notions and resources used to provide a categorical characterization of the natural number structure, can the moderate realist claim to achieve a firm grasp of *these* notions? Call this the *revenge challenge* for the moderate mathematical realist.

Button and Walsh (2018: 7.9) dispute the very possibility for success of a strategy that adopts a logic stronger than FOL to resolve the indeterminacy of arithmetical reference. The reason is that no logic that is compact will be able to categorically characterize ℕ. This means, they claim, that an "understanding of what consequence amounts to according to any such logic must, then, come from a specification of the formal semantics for that logic" (Button and Walsh 2018: 162). Yet, the notions involved in explicating the semantics of the kinds of logics needed far outstrip the resources of FOL and are thus, against a semantic theory formulated in FOL, "up for reinterpretation" (159). Thus, reference to the notions required to formulate the semantics of the logics needed to categorically characterize ℕ is inevitably as dubious, brute and mysterious as reference to the natural number structure was in the first place. Little has been gained by invoking stronger logics:

> it will always be at least as hard for moderate modelists to explain how they grasp the intended semantics for the logic in question, as it is for

9. Murzi and Topey (2021: n.16) remark that there is no essential difference between adopting SOL to solve the determinacy problem or any other logic using "essentially second-order notions," such as, e.g., (in)finitude. For reasons that will become clear below we disagree with this assessment.

> them to explain how to pin down the "standard model" of arithmetic in the first place. (Button and Walsh 2018: 162)
>
> Consequently, it is at least as difficult for moderates to explain how we can pin down those concepts, as it is for them to explain how we can pick out the isomorphism type of an $\omega$-sequence. (Button and Walsh 2018: 167)

The basis of Button & Walsh's negative assessment of the prospects of a realist strategy making use of a logic stronger than FOL is the insistence on the need for a semantic articulation of such logics. In doing so one will have to invoke mathematical notions determinate reference to which is as much in need of explanation as is reference to the natural number structure itself. The original question thus simply re-emerges "one level up," for one can ask in virtue of what determinate reference to *these* objects is achieved (Button and Walsh 2018: 164).

The claim that logics "strong enough to provide categorical theories of arithmetic must be articulated semantically" is based on the non-axiomatizability of these logics that results from their incompactness, coupled with the idea that, to have a determinate grasp of a notion, it must be "completely specified." There is, however, a different and arguably more natural way to conceive of what it means to have a determinate grasp of a notion. What is required to meet the revenge challenge is the provision of a characterization of the relevant notions of the logic together with a demonstration that this characterization suffices to *uniquely fix* them, that it, in other words, *implicitly defines* them. If no alternative interpretation is possible with respect to the provided characterization, the mathematical realist must be taken to have established a mechanism by which determinate reference (and thus the legitimacy of these notions in a description of the natural number structure) is assured.[10]

Nothing in the idea of a *categorical*, i.e., determinate, characterization of a notion demands that the logic in which it occurs be completely specified; such

10. The history of the relationship between notions of completeness and categoricity is fascinating and involved, see (Awodey and Reck 2002). In the present context, the crucial idea is that a loss of completeness does not amount to a loss of a determinate grasp, and thus the need for a semantic specification of a logic (which drove the argument of the model-theoretic sceptic) is not forced upon the mathematical realist. Rather, the claim to possess a determinate grasp of a notion can be maintained in the face of an incomplete specification as long as it can be established that this incomplete specification is *categorical* for the notion thus specified. The proposal, spelled out in more detail in the next section, is thus to replace a notion of completeness with a notion of categoricity. It was precisely the misalignment of completeness and full, i.e., categorical, determination that motivated Carnap (1943) to propose an alternative form of axiomatization of the propositional calculus and of FOL. He assigned to what we termed *Carnap-categoricity* in (Bonnay and Speitel 2021) an equally important status as was given to completeness, although tradition has not followed him in this.

completeness is not a necessary condition for its unique determination. Our understanding of a notion and the way it interacts with others might be partial and incomplete, but this might nevertheless suffice to uniquely determine it and thereby guarantee determinate reference. To meet the challenge of the sceptic that characterization has, of course, to be given in terms that are naturalistically acceptable and do not appeal to semantic notions with indeterminate reference. This is what we try to do in the next section.

## 3. Securing Arithmetical Determinacy

The question how we can determinately refer to the natural number structure has been "pushed upwards" and transformed into the question how we can determinately refer to the notions used in a categorical characterization of $\mathbb{N}$. The objection of the skeptic is the same as before: what warrants the exclusion of unintended interpretations of these notions? When it comes to the notion of *infinity*, doubts over its determinability stem from the fact that it is not first-order definable and its logic is not recursively axiomatizable. The point of contention thus concerns the question what it takes to "fully formalize" a notion such as the quantifier "there are infinitely many": can determinate reference to this quantifier be guaranteed without recourse to the notion of an $\omega$-sequence, the very notion we wish to determine in terms of it?

We will argue that the impossibility of recursively axiomatizing the logic of $\mathscr{L}(Q_0)$ need not constitute an obstacle for a determinate grasp of the notion of infinity. A determination of the meaning of an expression and a complete account of the truths involving it can come apart:[11] It is perfectly possible to uniquely pin down the model-theoretic value of a notion without requiring a prior determination of the full class of truths and consequences it gives rise to.[12]

11. Carnap showed the inverse of this claim long ago: he demonstrated that a complete account of the truths a notion gives rise to in the context of a logic is not a sufficient condition for its unique determination (Carnap 1943). Recently, building on work in (Bonnay and Westerståhl 2016), these investigations were extended to languages richer than FOL and it was demonstrated that unique determination of the meaning of a quantifier by a set of inferences is, in general, insufficient for axiomatizing the resulting logic (Westerståhl and Speitel 2022). Hence, a full grasp of the truths of a logic is neither necessary nor sufficient for the unique determination of the notions giving rise to it.

12. Similar observations can be found in Field's writings: "The incompleteness of formal arithmetic results from the incompleteness of the theory of … [the quantifier 'there are finitely many']; but if the latter is held determinate despite the undecidability of certain sentences in it, the same will hold derivatively of the concept of natural number" (Field 2001: 338). See also (Field 1980: ch. 9), (Field 1994), (Field 2001: ch. 11 & 12). We will discuss his approach further in §4.1.

### *3.1. Grasping (In)finitude*

Our moderate mathematical realist has been engaging in arithmetical reasoning and practice her entire life; first pre-theoretically by counting discrete objects in day-to-day encounters and later much more systematically through her work in a mathematics department. She has now been challenged by the model-theoretic skeptic to justify how her claims can be said to be *about* the intended structure $\mathbb{N}$, how she can be certain that this was the object she has been talking about all along when doing arithmetic.

During her early studies of arithmetic, e.g. in the context of a high school mathematics class, she established that there is no largest prime number; that, for every prime number $n$, there is a prime number $m$ that is larger. Asked *how many* primes there are she concluded that there must be infinitely many, for otherwise there would have to be a prime number $n$ such that there is no larger. However, when trying to express the claim "There are infinitely many natural numbers that are prime" in the language of arithmetic she had been operating with, the language of PA, she finds that this simple-seeming claim can only be expressed in a very cumbersome and unintuitive manner. This state of affairs is rendered even more unsatisfactory by the apparent grammatical parallel between statements of the form "There are $n$-many objects that have property $P$" and "There are infinitely many objects that have property $P$," the former of which possesses a formalization in FOL that faithfully mirrors its surface grammar, whereas the latter does not.

This is a somewhat unhappy situation, if only because it would be very nice to have a shorthand for saying that there are infinitely many natural numbers $n$ that possess a particular property $P$, this claim immediately entailing, for example, that there can be no largest natural number $m$ possessing $P$. It is furthermore a type of claim that our moderate mathematical realist encounters frequently in her arithmetical practice and uses regularly to communicate certain facts about collections of natural numbers without having to resort to cumbersome circumscriptions or paraphrases. To remedy this shortcoming she resolves to introduce an expression filling the linguistic gap discovered.

Note that she does not wish to talk about an object, *infinity*; she merely wishes to express that *there are infinitely many natural numbers that possess a particular property*, completely analogous to the claim that there are, say, 397 natural numbers that have a property $P$.[13] She thus does not set out to detail and investigate the properties of the object *infinity*, on par with the way she characterizes the object *71* (is the 20th prime number; is a permutable prime; is a twin prime, etc.), but, rather, treats the expression "there are infinitely many"

13. See (Field 1980: 93) for a similar observation.

as part of the framework used to talk about natural numbers and state properties of these, on par with the grammatically similar expression "there are at least $n$-many."[14] In other words, she treats the expression as a quantifier of the logical language used to talk about natural numbers, not as something denoting an object in the range of those quantifiers.

Being part of the logical superstructure for talking about natural numbers she can lay down rules for this quantifier. These state under what conditions we are warranted to infer that there are infinitely many natural numbers that $\varphi$ (e.g., when there is a natural number that $\varphi$'s and for every natural number that $\varphi$'s there is a larger one that also $\varphi$'s) and express what it means for there to be infinitely many natural numbers that $\varphi$, i.e., what we may infer from the claim that there are infinitely many natural numbers that $\varphi$ (e.g., that there is no largest natural number $n$ that $\varphi$'s).

Since the expression "there are infinitely many," $Q_0$, is regarded as belonging to the logical superstructure for the language of arithmetic, and not as part of the arithmetical lexicon itself, the moderate realist, in a first attempt, settles on a characterization of the central inferences involving the expression $Q_0$ that are free from arithmetic-specific vocabulary and assumptions, i.e., (i) and (ii) below:

(i) $\Delta_\varphi \vDash Q_0 x\varphi(x)$
(ii) $Q_0 x\varphi(x) \vDash \psi$ for all $\psi \in \Delta_\varphi$

where $\Delta_\varphi = \{\exists_{\geq n} x\varphi(x) \mid n \in \mathbb{N}\}$ and $\exists_{\geq n}$ is the FOL-definable quantifier "there exist at least $n$."[15]

Note that the moderate realist has, so far, said nothing about the model-theoretic meaning of $Q_0$ and thus, by omission, avoided the challenge of the sceptic. However, for the proof of categoricity to go through she won't be able to remain silent on this issue. Being pressed by the skeptic to explicitly state under what conditions a statement of the form $Q_0 x\varphi(x)$ is true in a model $\mathcal{M}$ her reply that $\mathcal{M} \vDash Q_0 x\varphi(x)$ iff $\mathcal{M} \vDash \psi$ for all $\psi \in \Delta_\varphi$ is deemed insufficient and rejected on the basis of compositionality considerations. The sceptic demands that the truth conditions be stated explicitly in terms of (properties of) the extension of $\varphi(x)$ and the moderate realist obliges by providing the following truth-clause for $Q_0$:

$$\mathcal{M} \vDash Q_0 x\varphi(x) \text{ iff } \omega \leq |\{a \in M \mid \mathcal{M} \vDash \varphi(a)\}|$$

14. That the latter possesses an object-language paraphrase reifying the notion "$n$-many" in PA, whereas the former does not, is inconsequential for her intention to introduce a linguistic device allowing her to directly express "infinitely-many" claims.

15. Note that the invocation of "$\mathbb{N}$" here does not amount to a presupposition of $\mathbb{N}$. It merely abbreviates that $\Delta_\varphi$ contains all statements of a particular shape that can be formed according to the grammar of FOL.

Equivalently, she identifies the model-theoretic meaning of $Q_0$ in a model $\mathcal{M}$ with domain $M$ with $\mathcal{Q}_0(M) = \{A \subseteq M \mid \omega \leq |A|\}$ and invokes the standard satisfaction clause for generalized quantifiers.

At this point, the challenge for the moderate realist re-emerges, for the skeptic now asks what it is that ensures her grasp of $\mathcal{Q}_0$. How is it that she manages to refer determinately to *it*, rather than some deviant, unintended $\mathcal{Q}^* \neq \mathcal{Q}_0$? Even worse, if the characterization of $\mathcal{Q}_0$ requires determinate reference to $\omega$, as the satisfaction clause suggests, then the categoricity result of the moderate realist begs the question: determinate reference to an $\omega$-sequence has been secured by presupposing $\omega$.

Here, however, the moderate realist has a successful reply: it is the acceptance of the inferences (i) and (ii) that ensures determinate reference to $\mathcal{Q}_0$, for $\mathcal{Q}_0$ is the unique value for $Q_0$ rendering these inferences valid—there simply is no other, alternative value we could mean when reasoning in accordance with (i) and (ii). In other words, (i) and (ii) *uniquely determine* $\mathcal{Q}_0$—no prior recourse to $\omega$ is needed to achieve determinate reference to $\mathcal{Q}_0$ and the sceptical challenge thus fails to gain traction.

Let $\mathscr{L}(Q)$ be the language of FOL with an added quantifier-symbol $Q$. Let $\mathcal{Q}'$ and $\mathcal{Q}^*$ be (type-appropriate) interpretations for $Q$.[16] We say that $\varphi$ is true in a model $\mathcal{M}$ under the $\mathcal{Q}$-interpretation, $\mathcal{M} \vDash^{\mathcal{Q}} \varphi$, if $\mathcal{M} \vDash \varphi$ when $Q$ is interpreted by $\mathcal{Q}$. $\varphi$ follows from $\Gamma$ under the $\mathcal{Q}$-interpretation, $\Gamma \vDash_{\mathcal{Q}} \varphi$, if, for all $\mathcal{M}$, whenever $\mathcal{M} \vDash^{\mathcal{Q}} \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M} \vDash^{\mathcal{Q}} \varphi$. Then:[17]

**Theorem 1** Suppose that

(i′) $\Delta_\varphi \vDash_{\mathcal{Q}'} Qx\varphi(x)$
(ii′) $Qx\varphi(x) \vDash_{\mathcal{Q}'} \psi$ for all $\psi \in \Delta_\varphi$
(i*) $\Delta_\varphi \vDash_{\mathcal{Q}^*} Qx\varphi(x)$
(ii*) $Qx\varphi(x) \vDash_{\mathcal{Q}^*} \psi$ for all $\psi \in \Delta_\varphi$

Then $\mathcal{Q}' = \mathcal{Q}^*$.

The theorem establishes that the two patterns of inference (i) and (ii) *implicitly define* $Q$. Any two quantifiers interpreting $Q$ and obeying (i) and (ii) necessarily

16. The following set-up can be found in (Speitel 2020) and (Speitel and Westerståhl 2022) where it is used to explore a more general phenomenon, an instance of which we consider above. It is based on (Bonnay and Westerståhl 2016).

17. This is a special case of a much broader phenomenon, termed *Carnap-categoricity* in (Bonnay and Speitel 2021), and more systematically investigated in (Speitel 2020) and (Speitel and Westerståhl 2022). In fact, any $EC_\Delta$-definable quantifier is uniquely determined in this way. The results mentioned here can also be found in (Speitel 2020) and (Speitel and Westerståhl 2022). Their proofs are reproduced in the Appendix. The Carnap-categoricity of the quantifier "there are infinitely many" was first observed by Dag Westerståhl and is stated and generalized in (Speitel 2020).

coincide—the inferences constrain the space of admissible interpretations so tightly, there is only a unique denotation that "fits." Combining this with the fact that $\mathcal{Q}_0$ is consistent with (i) and (ii) (see Theorem 2 in the Appendix), this result demonstrates that our hold of $\mathcal{Q}_0$ is ensured by (i) and (ii). The inferential practice involving the quantifier "there are infinitely many" is sufficiently rich to ensure that our use of an expression for that quantifier does not admit any alternative, unintended interpretations. Determinate reference to $\mathcal{Q}_0$ is thereby guaranteed. Determinate reference to $\mathbb{N}$ can then be successfully achieved by means of the theory $\mathrm{PA}^+ = \mathrm{PA} + \{\forall x \neg Q_0 y(y < x)\}$.

Note that the logic of $\mathscr{L}(Q_0)$, in its intended interpretation, is not recursively axiomatizable. However, inferring according to (i) and (ii) suffices to "fix" a determinate meaning for $Q_0$. Thus, we can possess an implicit, yet full, understanding of $\mathcal{Q}_0$ so long as we infer in accordance with (i) and (ii), even in the absence of a complete and recursive axiomatization of the logic of the relevant notion, *contra* Button & Walsh's contention (Button and Walsh 2018: 7.9).[18] Recursive axiomatizability is thus not a good criterion for possessing a determinate grasp of a notion.[19]

The defense of a determinate grasp of $\mathcal{Q}_0$ on the basis of the inferential patterns (i) and (ii) might, nonetheless, ring hollow to the naturalistically minded realist. The reason for this lies in the infinitary nature of (i) and the accompanying need for naturalistically unacceptable inferential powers required by its utilization.[20] Without providing a way of rendering (i) naturalistically digestible the realist must be taken to have transgressed the limits of moderation in the assumption of non-computational inferential capacities. Her defense of $\mathcal{Q}_0$ is thus still incomplete. In the next section, we try to fill the argumentative gap that remains to defend the claim that $\mathcal{Q}_0$ is, in fact, naturalistically acceptable.[21]

### *3.2. Naturalistically Acceptable Determinacy*

Based on an idea of Feferman (2015), it was argued in (Bonnay and Speitel 2021) that the fact that $\mathcal{Q}_0$ can be uniquely determined by a consequence relation including the inferences (i) and (ii) is sufficient to render it a *logical* notion. Here I do not wish to go as far, for what is at issue for the moderate realist is not the *logical* standing of $\mathcal{Q}_0$, but the question as to its naturalistic acceptability on the basis of the inferential patterns (i) and (ii). Nonetheless, on the current approach

18. For a similar observation see (Murzi and Topey 2021: 23). For the claim that the notions of incomplete logics escape our determinate grasp, see sources in (Murzi and Topey 2021: n.45).

19. I am grateful to an anonymous reviewer of this paper for emphasizing this point.

20. Cf. the *Cognitive Constraint* of (Warren and Waxman 2020: 485).

21. I am grateful to two anonymous reviewers of this paper to put pressure on this important point.

the quantifier "there are infinitely many" is considered part of the logical superstructure used to formulate a (referentially determinate) theory of arithmetic. Our grasp of the notions that form part of this superstructure is mediated through our inferential uses of them. The crucial question thus becomes: can the sort of inferential use required to fix the meaning of $Q_0$ be made naturalistically acceptable in the contexts relevant for the moderate realist?

The central point of the above defense of using a stronger logic to categorically characterize $\mathbb{N}$ was that to possess a determinate grasp of the quantifier $Q_0$ it is not necessary to have a prior grasp of, or access to, an $\omega$-sequence given independently of the use of $Q_0$. Rather, accepting certain inferences governing the "intuitive," pre-theoretical behaviour of $Q_0$ ensures that reference is thereby uniquely fixed. The referent thus determined must be the intended interpretation of $Q_0$, because it is the only interpretation consistent with its inferential characterization. Note that the inferences laid down for the quantifier $Q_0$ are not grounded in properties of an independently assumed object, such as $\omega$, but rather concern the need for a logico-linguistic device to be able to state properties of certain progressions and collections of natural numbers.

However, even if the direct presupposition of $\omega$ could be avoided through recourse to inferential patterns (i) and (ii), this only shifts the challenge. To make the approach feasible for the moderate realist it must be established that (i) and (ii) constitute a *naturalistically acceptable* way of fixing the reference of $Q_0$. And the infinitary nature of (i) raises serious doubts about this assumption.

For while the infinity of the set of inferences of type (ii) seems unproblematic—in every particular case the actual inference is finite and thus, in principle, performable—this is not the case for inferences of type (i). Here it appears *in principle* impossible for cognitively finite agents like us to perform the inference as such performance would require one to go through infinitely many premises.[22] To maintain that an inference must be performable without assuming non-computational deductive abilities amounts, in the current context, to a *compactness-demand*: whatever can be inferred on the basis of a premise-set $\Gamma$ should already be inferable on the basis of a *finite* premise-set $\Sigma \subseteq \Gamma$ in order for the inference from $\Gamma$ to its conclusion to be performable. Only such compact inferences ought to count as feasible and are thus naturalistically acceptable.

Apart from arguing that performing infinitary inferences simply *is* possible under certain circumstances the moderate realist could at this stage point out that it need not be recognized, or even be recognizable, that a particular inference is being performed in order to reason according to it. Determinate reference, she

22. For an argument that (some) infinite inferences are feasible see (Warren 2020; 2021). For arguments that non-compact inferences are neither artificial nor unusual, see (Griffiths and Paseau 2022: ch. 5.3) and (Paseau and Griffiths 2021).

might claim, is achieved by inferring *in accordance with inferential patterns of the form (i)*, i.e., by not violating the pattern in one's reasoning, rather than by actually performing the inference. However, such a defense is bound to leave the case for the determinacy of arithmetic on the basis of adopting $\mathscr{L}(Q_0)$ in a worse state than the moderate realist might hope. Moreover, a more robust strategy is available to her.

For she can point out that the relevant contexts for performing an inference of form (i) are, for her purposes, *arithmetical contexts*. In these contexts, she can legitimately avail herself of arithmetical vocabulary in stating inferential patterns involving $Q_0$. In particular, she can replace the infinitary inference (i) by the, less general but finite, inference[23]

(iii) $LO_{\varphi(x)}(R) \vDash_Q Qx\varphi(x)$

where $LO_{\varphi(x)}(R)$ is the sentence that says that $R$ is a left-minimal, right-unbounded, strict linear order of the $\varphi$'s.[24] (i) and (iii) are equivalent over models with appropriate signatures (see Theorem 3 in the Appendix), and the assumption of (ii) + (iii) thus also suffices to uniquely determine $Q_0$. Hence, in the context of the language of arithmetic the moderate realist is able to determine the intended meaning of $Q_0$ in a naturalistically acceptable fashion.[25]

23. Note that the use of arithmetical vocabulary, and content-specific vocabulary in general, puts into doubt the status of $Q_0$ as a purely logical notion. This does not affect the point of the moderate mathematical realist however, for her goal is only to show how naturalistically acceptable determinate reference to $Q_0$ is possible, not that it is possible by means of logic alone. The original proof of $Q_0$'s Carnap-categoricity by Dag Westerståhl made use of a different signature and characteristic inference-pattern but proceeded analogously otherwise.

24. The possibility of giving a compact presentation of this inference in the context of richer vocabularies is one of the essential differences between the approach to arithmetical determinacy in terms of $\mathscr{L}(Q_0)$ as pursued here, and the utilization of an $\omega$-rule to achieve a categorical characterization of $\mathbb{N}$. A further advantage stems from the fact that proponents of $\mathscr{L}(Q_0)$ interpret a notion new to FOL, whereas the $\omega$-rule *reinterprets* a notion, the universal quantifier, already present as well as assumed understood and determined by its usual rules. For this reason we think the approach in terms of $\mathscr{L}(Q_0)$ is preferable to the use of an $\omega$-rule in establishing the determinacy of arithmetic.

25. The result makes use of the fact that the semantics for generalized quantifiers adopted here are such that the extension of $Q_0$ over a model $\mathcal{M}$ is a function of its underlying *domain*, that, in other words, the interpretation of a quantifier only depends on the underlying set. This assumption can be motivated in several ways: Assuming permutation- or isomorphism-invariance for quantifiers—on the basis of their relation to quantity/cardinality or, alternatively, considerations of logicality (see, e.g., (Tarski 1986), (Sher 1991))—suffices to force the interpretation of $Q_0$ to be identical across identical domains. Even less demanding, assuming the standard interpretation of the usual logical constants of FOL entails the fixity of $Q_0$'s interpretation across identical domains. Nevertheless, there is a genuine question regarding how legitimate it is for the moderate realist to appeal to and adopt any of these assumptions. I am grateful to an anonymous reviewer of this paper for raising this issue for the moderate realist.

A further revenge objection looms large, for in the compact characterization of $\mathcal{Q}_0$ via (ii) + (iii) extensive use was made of arithmetical concepts (e.g., linear orders). This, the model-theoretic sceptic complains, is illegitimate as it amounts to using notions and concepts in determinately fixing the meaning of $Q_0$ whose very determinacy, ultimately, depends on $\mathcal{Q}_0$. We can here fruitfully distinguish between the positions of the naïve and the refined sceptic. The naïve sceptic holds that the concepts used in achieving determinate reference to $\mathbb{N}$ must be completely understood/understandable in all their consequences. This, I argued above, is unnecessary to possess a determinate grasp.

The refined sceptic, on the other hand, assumes that the notions we use to characterize the natural number structure must be fully determinate *before* we can use them towards this goal. This, I think, is equally mistaken. The determinacy of a notion, and the determinacy of theories making use of that notion, can develop in tandem, mutually precisifying and fixing each other through their interaction. We might have to use a moderate amount of first-order describable arithmetic to state essential properties of the quantifier "there are finitely/infinitely many" which we can then, in turn, use to characterize the (models of the) theory we applied to describe those properties. Each component by itself might inherit indeterminacy from the other. In conjunction, however, they mutually ensure determinate reference:[26] "we needn't first secure the determinacy of a concept before we use it in reasoning: if that were required, reasoning could never get started" (Field 2001: 342).

In using the theory of linear orders to describe the conditions under which we might infer a sentence involving $Q_0$ we are leaving the particular kind of linear order we have in mind underdetermined. Despite this underdetermination, it suffices to fix the intended meaning of $Q_0$ which we then use to specify the nature of the relevant linear order further. The objection thus, once again, relies on additional, unjustified assumptions regarding the process of fixing reference.

## 4. Alternative Proposals

Having argued for a response to the sceptical challenge on behalf of the moderate mathematical realist in the previous section, I will now briefly consider two alternative proposals for securing determinate arithmetical reference: Field's expansion of arithmetical practice (Field 1994; 2001), and Murzi & Topey's (2021) defense of the categoricity of the second-order quantifiers.

26. See also (Field 1994: 396): "To say that we do not have a determinate understanding of something is not to say that we have no understanding of it at all."

### *4.1. Field: Inferential Practice and Mathematical Applications*

The question of the model-theoretic skeptic is "[w]hat are the limits of the semantic facts that our inferential practice might determine?" (Field 2001: 339). The answer provided, by invoking non-standard models, is that those limits fall short of determining all relevant facts concerning the notion of finitude. Field claims that this is due to—artificially and unnecessarily—restricting ourselves to the inferential practice of pure mathematics, thereby ruling out further resources we have at our disposal for fixing the content of concepts. Just as there are non-formal causal or phenomenal constraints that help us fix the reference of "planetary body" or "cat," he maintains that we can avail ourselves of additional constraints on permitted models, thereby ruling out some unintended candidates.

In the case of mathematics Field advocates for broadening our conception of the inferential practice we take to contribute to fixing reference of mathematical expressions. Relevant additional constraints on permitted models could stem from *physical applications* of the mathematical apparatus.[27] Given that observational or, more broadly, non-mathematical scientific vocabulary can be fixed by non-mathematical constraints, that unintended models for these notions can be ruled out on the basis of mechanisms that are not subject to a revenge-application of the "just-more-theory"-manoeuvre, we are thus supplied with additional means to restrict the extensions that mathematical vocabulary used in the formulation of theories may take on.

More concretely, the hope is that "*if the physical world is as we typically think it is*, our physical beliefs are enough to determine the extension of 'finite'" (Field 1994: 416). The precise nature of the physical assumptions Field considers for fixing the extension of "finitely many" is unimportant for present purposes.[28] The "key to the argument … is the assumption that the physical world provides an example of a physical $\omega$-sequence that can be determinately singled out" (Field 2001: 341). Since access to this $\omega$-sequence proceeds via physical vocabulary, which is not subject to the same referential indeterminacy affecting mathematical vocabulary, it constrains reference of our mathematical vocabulary in a non-objectionable way: "the constraints on the physical vocabulary determine a privileged class of interpretations in which 'finite' determinately stands for what it should" (Field 1994: 417). Expanding the inferential practice to be taken into account in fixing the notion of "finite" to include applications of mathematics to physical theory thus yields an inferential practice rich enough to rule out non-standard interpretations of the quantifier "there are finitely many."

27. See (Field 1994: 414–420) and (Field 2001: 340–342). See also (Weston 1976: 296–297) for a similar argument.

28. See (Field 1994: 416–420) and (Field 2001: 340–342). Cf. also (Parsons 2007: 289–290) and (Warren and Waxman 2020).

Our approach has much in common with Field's, including the idea that knowledge of concepts such as "(in)finitely many" is mediated through our inferential use of them. It nonetheless differs in an important respect: Field leaves the inferential practice of pure mathematics and incorporates inferential practices of the (applied) sciences that use mathematics. The approach of §3, while potentially having to move beyond purely logical practices,[29] remains squarely within the inferential practice of mathematics itself, with no recourse to physical or observational notions of any kind.

This makes a difference: while Field's approach requires a prior grasp of a particular and special *physical object*—an instantiated $\omega$-sequence on the basis of which we are able to obtain a determinate notion of "finite set" by means of which we can then appropriately state the semantics of the quantifier "there are finitely many"—the approach pursued above deems the existence of a (partial) characterization of the inferential behaviour of the quantifier "there are infinitely many" sufficient to determine its meaning.[30]

Thus, on Field's account, to achieve determinacy of reference for arithmetical notions we have to accept several strong empirical assumptions concerning the infinitude of space or time.[31] It is this dependence on empirical fact—Field's *cosmological assumption* concerning the structure of the world (Field 2001: 340)—that has been most frequently criticized:

> I find it hard to see how someone could accept that [cosmological] assumption who does not already accept some hypothesis that rules out nonstandard models as unintended on mathematical grounds. If our powers of mathematical concept formation are not sufficient to do the latter, then why should our powers of physical concept formation do any better? (Parsons 2007: 290)

"[U]sing the physical world to explain mathematical determinacy" risks getting "the intuitive conceptual priority … backwards" (Warren and Waxman 2020: 490), thereby leaving the determinacy of the notion "finitely many" in an

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29. On the account of logicality developed in (Bonnay and Speitel 2021) "there are infinitely many" qualifies as a logical notion. If one accepts the inferential patterns (i) and (ii) above as legitimate for determining meaning we do not even need to invoke inferential practices outside of logic itself, though see the possible objections discussed in Section 3.2.

30. Note the different mechanisms of fixing "(in)finitely many": On the proposal of §3, it is inferential patterns involving the term $Q_0$ that determine its meaning "directly." On Field's account, our broad inferential practice determines the notions "set" and "$\in$" well enough to allow us to *define* the quantifier "there are finitely many" (Field 2001: 339).

31. Field emphasizes that there are a variety of empirical hypotheses that would secure the standard meaning of "finitely many," and that his "cosmological assumption" is just one possible candidate among many.

unstable state.[32] The account proposed in §3 offers an alternative to fixing the meaning of the expression "there are infinitely many" that introduces new inferential patterns to take into account, but does not rely on a prior physical basis for its determinacy.[33]

### *4.2. Murzi & Topey: Categoricity by Convention*

The by far most popular response to the non-categoricity of PA consists in a move to a second-order framework and to consider $PA_2$ instead. *Dedekind's Categoricity Theorem* then ensures the categoricity of arithmetic. However, just as popular as this line of response to the underdetermination of arithmetic are criticisms of approaches utilizing second-order quantification to guarantee determinacy (see (Weston 1976), (Field 2001), (Shapiro 2012), (Meadows 2013)). The main objection to the use of SOL in establishing the categoricity of $PA_2$ relates to the assumption of a *full* semantics for second-order logic. For here the model-theoretic skeptic asks in virtue of what the proponent of SOL can claim to be using a full rather than a Henkin-semantics (see §2.3).

The success of the project of using SOL to achieve determinate reference to the natural number structure thus stands and falls with the ability to ensure that the semantics for the second-order quantifiers is full. And the crucial question here is: in virtue of what is it that the proponent of SOL can reasonably claim that her second-order quantifiers range over the full power-set of the domain, rather than an inferentially indistinguishable subset thereof.[34]

Murzi and Topey (2021) defend the claim that the proponent of SOL is indeed using a full, rather than a Henkin-semantics. Their core contention is that it is not the concrete rules and axioms of any particular formal theory that fix the meaning of the expressions occurring in them, but that it is the *dispositions to infer* underlying those rules which ground their determinate semantics.

32. Field himself remains somewhat dissatisfied: "it is not entirely attractive, but I know none that is better. … I am sure that some will feel that making the determinateness of the notion of finite depend on cosmology is unsatisfactory; perhaps, but I do not see how anything *other* than cosmology has a *chance* of making it determinate" (Field 1994: 416, 418).

33. Cf. also (Carrara et al. 2016) for an alternative response based on the possession of a primitive notion of finiteness.

34. Field (1994: 396) points out that aspects of the inferential practice of SOL allow us to go beyond Henkin-semantics. Nonetheless, even a generalized conception of inferential practice that suffices to go beyond Henkin-semantics still falls short of establishing a full semantics. Since nothing essential depends on this here, we will, for presentational purposes, not consider any of these non-full but richer candidates, and continue to phrase the main difference as a difference in choice between Henkin- and full semantics.

On their account, the dispositions to infer in accordance with a particular set of rules are not exhausted by concrete rules for particular languages or language-fragments. Rather, the particular rules are accepted *because* they are manifestations of these dispositions when considering a concrete language. The fact that rules are grounded in dispositions to infer means that we are "disposed to accept all instances of those rules irrespective of how we expand our language" (Murzi and Topey 2021: 3). This, in turn, means that it is best to understand rules not as particular collections of their instances with respect to a concrete vocabulary but, rather, as what Humberstone (2011) calls *language-transcendent*, i.e., as functions from languages to collections of instances over these languages, or, following McGee (1997; 2000; 2006), as *open-ended*, i.e., as continuing to hold unrestrictedly under expansions of the language. It is the open-endedness of rules, grounded in language-transcendent inferential dispositions, that fixes the full semantics of the second order quantifiers: "the open-endedness of our inferential dispositions … suffices to yield full second-order logic" (Murzi and Topey 2021: 1).

Dispositions to infer remain undisturbed by the language in which the concrete inferences take place. The robustness under expansions or switchings of the language constitutes the open-endedness of rules of inference manifesting these dispositions. A rule is *open-ended* if it continues to hold, if it remains valid, under the addition of new vocabulary to the language. To account for the extreme generality of dispositions to infer the semantics of the expressions occurring (non-schematically) in the rules of inference must be able to guarantee the continued validity of these rules no matter the precise language in which they are instantiated. This affects, in particular, the rules for the second-order quantifiers. Assuming that every object in the domain of discourse of a model is *in-principle* nameable forces the range of the quantifiers to be maximal, thereby ensuring a full semantics.

Let us reconstruct the argument in a bit more detail: Because our dispositions to infer are unrestricted, rules of inference must be open-ended – they must continue to apply irrespective of any particular language. Assuming that "no item is in principle unnameable in any language that expands our own, guarantees that the logical rules hold universally: for any item whatsoever, there exists an expansion of our language in which there's a term naming that item, and the logical rules hold in all such expansions" (Murzi and Topey 2021: 19).[35] The open-endedness of the rules, combined with the in-principle nameability of every object in the domain, guarantees that the interpretations of the quantifiers (first- and second-order) must be permutation-invariant. Generalizing the result

35. Note that skepticism about the feasibility of such languages, questions concerning how it would be possible to add terms to a language naming every object we like, etc., are misplaced at this point. All Murzi & Topey's argument requires is that every such expansion constitutes a *possible language* (Murzi and Topey 2021: 19, 23).

by (Bonnay and Westerståhl 2016) that permutation-invariance forces the first-order quantifiers to take on their standard interpretation to the second-order case,[36] Murzi & Topey then establish that the only semantics consistent with the open-ended understanding of the rules for the second-order quantifiers is a full semantics.[37] Reasoning in accordance with our dispositions to infer thereby secures a full semantics for second-order logic and rules out Henkin-semantics as viable alternative—the indeterminacy of second-order semantics is thus removed.[38]

### 4.3. *Some Issues for the Second-Order Strategy*

Murzi & Topey propose an attractive strategy to secure a full semantics for the second-order quantifiers and, thereby, determinate reference to the natural number structure. They provide a diagnosis of the indeterminacy of the second-order quantifiers that is continuous with a phenomenon already found in the first-order case and propose a unified mechanism for fixing the semantics for the expressions of FOL and SOL.

Both their and our diagnosis of the mistake in the sceptic's argument maintains that the sceptic fails to appropriately capture what is necessary to possess a determinate grasp of a notion (cf. (Murzi and Topey 2021: 26)). Similarly, they say that "speakers (and theorists) need not grasp the complex mathematical concepts involved in the standard semantics for SOL in order for it to be guaranteed that SOL is correctly interpreted by that semantics" (Murzi and Topey 2021: 26).

36. Murzi & Topey observe that the underdetermination of the second-order quantifiers is a special case of a much wider underdetermination phenomenon affecting also the first-order quantifiers. That the first-order quantifiers admit of non-standard interpretations consistent with their usual rules of inference was first observed by Carnap (1943). Bonnay and Westerståhl (2016) later characterized the degree and shape of this underdetermination precisely and showed that it can be remedied by demanding that the interpretations of the universal and existential quantifiers of FOL be permutation-invariant.

37. I am skeptical about the claim that "this result remains available even independently of our inferentialist assumptions, on the alternative (widely accepted) assumption that permutation invariance is necessary for logicality" (Murzi and Topey 2021: n.49). This is due to the way permutation-invariance is put to use in Murzi & Topey's argument, i.e., as applying directly to second-order objects (see (Murzi and Topey 2021: n.37)) rather than via permutations of the underlying first-order domain as in more orthodox approaches to logicality (see, e.g., Tarski (1986)). I will not pursue this issue further here.

38. The significant influence of constant-terms on possible interpretations of quantifiers has also been observed in the case of non-standard interpretations of the first-order quantifiers; see (Antonelli 2013: 638/639): "singular terms are unconstrained in their taking denotations […], thereby giving access to the 'dark corners' of the first-order domain where the light of the quantifiers does not shine (i.e., less figuratively, allowing reference to objects that fall outside the range of the quantifiers)."

It is sufficient to infer in accordance with the inferences characteristic of a notion, or to exhibit the right kinds of dispositions to infer, to ensure that the notion in question possesses its intended meaning.

Nonetheless, one might wonder how innocent the notion of *open-endedness*, which does much of the work in Murzi & Topey's argument, really is. In particular, is the invocation of this notion tantamount to presupposing an understanding of full second-order quantification? If so, little is gained by its use in defending a full semantics for SOL. But even if not, there is the worry that it invites a "just more theory"-type revenge objection, in that by talking about possible extensions of a language we are (implicitly) quantifying over all possible languages, where this kind of quantification is as much in need of explanation as second-order quantification itself. I don't believe this objection has as much force as some of its proponents have taken it to possess. The crucial idea, when it comes to open-endedness, is to reason in accordance with a rule *come what may*. This "come what may"-part, however, appears to place far fewer demands on our grasp of the notion of open-endedness than does quantification over all possible languages. What we need to know and understand is the general *shape* a language can take, but this falls short of requiring a grasp of the *totality* of all possible languages.[39]

There is, however, a further worry concerning the need for open-ended *logical* principles over and above open-ended *mathematical* principles. This line of criticism can be interpreted as questioning the need for a move to a second-order framework, plagued by all the usual worries regarding its ontological commitments, logical status, etc., given that its advantages,—determinate mathematical reference,—can already be had in a first-order setting. The central insight here is that the first-order induction-schema, properly understood as open-ended, already suffices to achieve categoricity without requiring the additional use of second-order quantifiers. For, properly understood, the schema says that induction holds for *all* properties on the natural numbers. Its limitations stem from the restrictive interpretation of "property" in the usual setting of first-order PA and an understanding of the schema as a meta-language abbreviation for an infinite set of object-language sentences. This, however, arguably misrepresents its intended scope and meaning. Thus, the unrestricted nature of induction might be better captured by directly using schematic variables in the object language.[40] The use of such object-language schematic expressions, the argument goes, best expresses the generality and open-endedness of induction

39. So it seems to me, at least. Though see (Field 2001: Postscript) for labelling this type of response a *cheat* and (Pedersen and Rossberg 2010) for the hidden ontological commitments of open-endedness. Cf. also (Walmsley 2002: 253).

40. See, e.g., (Feferman 1991: 2).

without incurring the ontological commitments that go along with explicitly quantifying over these variables.[41]

The schematic interpretation of the induction schema, as a way of expressing its unrestrictedness explicitly in the object-language itself, is sufficient to establish the categoricity of arithmetic in the *free-variable fragment of second-order logic*, i.e., SOL without second-order quantifiers but permitting the occurrence of second-order variables.[42] Not only is the assumption of second-order quantifiers ranging over the full powerset of the domain therefore not needed to achieve categoricity, many of the (perceived) disadvantages of the use of full second-order logic in the characterization of the natural number structure can thus also seemingly be avoided: a proper implementation of open-endedness in the form of the schematic interpretation of the induction schema yields categoricity "on the cheap."

Leaving the ontological costs of schematic induction as against second-order quantified induction undecided, one advantage that has been attributed to the free-variable fragment of second-order logic is that it steers clear of issues pertaining to the non-robust nature of the power-set operation that features so prominently in the full semantics of the second-order quantifiers. If, as has often been claimed, we are lacking a clear understanding of the notion "all subsets of" when it comes to infinite domains, i.e., of the power-set operation, this lack of clarity immediately affects our understanding of full second-order quantification itself. Note that the open-endedness defense advanced by Murzi & Topey is of no help here, as we can grant that whatever interpretation our second-order quantifiers must have if our dispositions to infer are unrestricted must be full. Yet, since it is indeterminate what it means for an interpretation to be full, due to the non-robust nature of the set-theoretic operation used to spell out the notion of *fullness*, the fact that the appropriate semantics for the second-order quantifiers must be a full semantics does little to remove the indeterminacy inherited through this "background indeterminacy" (see (Lavine 1994: 237/238)).

None of this seems to affect the understanding of the schematic version of induction:[43]

> one can work with theories formulated in free-variable second-order languages, and one can coherently maintain the categoricity of arithmetic and analysis, without claiming some sort of absolute grasp on the range

41. See (Shapiro 1991: 247/248) and (McGee 1997: 60). See (Pedersen and Rossberg 2010) for criticizing this move as an attempt to illegitimately hide ontological commitments.

42. See, e.g, (Shapiro 1991: 248) and (Corcoran 1980).

43. See also (Lavine 1994: 231): "we understand when a given expansion of a language is permissible. But that is a far cry from the commitment required for using second-order universal quantification—the commitment to a determinate but unexplained fact of the matter about what *all* the subcollections of a domain are."

> of the relation and function variables—or even claiming that there is a fixed range. One only needs the ability to recognize subsets as they are defined, and in the context of the interpreted formal languages in question this is not problematic. This is a rather weak hold on the range of the second-order variables. (Shapiro 1991: 247–248)

It is not difficult to see that the free-variable fragment of second-order logic is equivalent to only allowing initial universal quantifiers over second-order variables (see (Shapiro 1991: 246–250) and (Corcoran 1980)). For this reason it has been described as a mere notational variant of a fragment of SOL, affected by the same sorts of problems pertaining to the semantics, ontological commitment, and indeterminacy of SOL.[44] Whatever the precise relation between SOL and its free-variable fragment, however, their difference prompts some questions for the way Murzi & Topey wish to secure the determinacy of arithmetical reference.

The question is why we should want to invoke second-order quantification at all. Given the success of the free-variable fragment, why not agree with the dispositional story concerning the open-endedness of genuine arithmetical claims (i.e., induction), but stop there? What need is there to invoke further *logical* notions when the open-endedness of the mathematical claims of the theory under consideration suffice to determinately fix reference? Why, in other words, assume more than is needed for the goal at hand? This point is not so much an indication of a shortcoming of the position, but a request for a justification of the additional logical machinery employed. It is a significant request, because the invocation of that machinery under its preferred interpretation brings with it an interesting further type of indeterminacy, seemingly absent in the free-variable fragment: the non-robustness of the power-set operation needed to spell out the intended semantics. Let me elaborate further on this point.

While the indeterminacy of the background theory of models, engendered by the possibility of non-standard models of set-theory, affects all model-theoretic responses to the skeptical challenge, the indeterminacy "cuts deeper" in the case of second-order logic. The existence of Löwenheim-Skolem models of the background set-theory affects the determinacy of the entire model-theoretic apparatus that provides the space of meanings for the notions under consideration. But there are further types of indeterminacy affecting specifically second-order quantification. I will address the former type of indeterminacy first.

In any first-order framework Löwenheim-Skolem models are as unsatisfactory as they are unavoidable. They create an inescapable level of indeterminacy

44. See, e.g., (Button and Walsh 2018: 162–163) and (Field 2001: 355). See (Lavine 1994: 229–240) for a defense.

for any defensible realist position as conceived in the context of this paper.[45] Consider, for example, the second-order quantifiers. Provided they are understood in terms of a set-theoretic semantics and we somehow make sure that a correct interpretation of them is a full semantics this still does not completely remove any possible indeterminacy. This is because what it means for an interpretation to be full might differ between different models of the background set-theory used to spell out the notion of a full interpretation. The existence of non-standard models of first-order ZFC, however, is guaranteed by the *Löwenheim-Skolem Theorems*. As a consequence, the notions needed to spell out the content of "full interpretation" and, as a result, that notion itself, remain indeterminate.[46] All Dedekind's Categoricity Theorem is able to provide is thus an assurance that *once we have fixed an interpretation of the background set-theory*, and thereby settled the meaning of, for example, "all subsets," all models of the target-theory will be isomorphic. Analogous considerations apply to the notion "infinitely many."

Assuming the standard model of set-theory outright is certainly much less warranted than assuming the standard model of arithmetic. However, making some moderate assumptions about the *shape* of acceptable models of the background set-theory helps reduce the unconstrained indeterminacy caused by Löwenheim-Skolem type phenomena significantly. These assumptions amount to supposing that the models of set-theory we are dealing with are *transitive models*.[47] Call these *acceptable models* of the background theory.[48] What we remain ignorant about when dealing with an acceptable model concerns the true extent of the set-theoretic universe.

A notion that is insensitive to the extent of the set-theoretic universe, i.e., that is such that it does not change its meaning depending on the actual extent of the universe of sets, is an *absolute notion* (see, e.g., (Jech 2003: 163)). A *non-absolute* notion is one whose meaning depends on the extent of the underlying universe of sets in a particularly crucial way: it changes its meaning over fragments of the set-theoretic universe depending on how those fragments are expanded or restricted—its meaning is intrinsically unstable. Here we are specifically concerned with notions absolute among *acceptable models* of ZFC.

45. The criticism that background indeterminacy remains is not new; see, for example, (Weston 1976), (Parsons 1990), (Field 2001: Postscript), (Button and Walsh 2018: 158) and (Button 2022).

46. See, e.g., (Field 2001: Postscript), (Button and Walsh 2018: 158), and (Button 2022).

47. See, e.g., (Jech 2003: 163) for a precise definition. Roughly, this amounts to assuming that the membership relation of the model is the real membership relation and not some inaccurate representation of it.

48. Cf. (Field 1994: 416) for the recognition of the importance of *transitive models of set-theory* for the question of referential determinacy.

While a property such as *being finite* is absolute (among transitive models of ZFC) (see, e.g., (Kunen 2011: 123)), several set-theoretic notions crucial to the full semantics of SOL, such as, e.g., the power-set operation, are not. SOL's demands on the background universe of sets are thus much more substantial.[49] This has consequences for the set-theoretic robustness of second-order consequence more generally.[50] Note that the instability of non-absolute notions is a very different sort of indeterminacy than the one that arises due to the existence of non-standard models of the background set-theory which, in some sense, cuts deeper. The latter is a result of the inability to rule out deviant models of the background set-theory due to expressive limitations of first-order languages. The former shows that even if the notions "is a set" and "$\in$" are not thoroughly mis-interpreted, as long as their interpretation is left partial, indeterminacy can arise. Thus, even among acceptable models of the background set-theory non-absolute notions are affected by indeterminacy.

The finite/infinite distinction turns out to be set-theoretically robust in this sense: the notion of finitude is *absolute* (among transitive models of ZFC). This means it is, to a certain degree, independent of the real, underdetermined set-theoretic universe in that transitive models of set-theory will not disagree on which sets to count as finite. No matter what transitive model of set-theory we are operating in, then, i.e., no matter the actual extent of the set-theoretic universe, the notion "finitely many" will not admit multiple, divergent interpretations in them. What is underdetermined is what is the *actual* extent of the universe of sets, but not the meaning of "finitely many" in it, whatever it may be. The intended semantics for second-order logic, on the other hand, is deeply affected by this additional, intermediary type of indeterminacy and it is unclear how to rule it out short of simply postulating a primitive understanding of what it means to be a *full* semantics.

This demonstrates that the indeterminacy of the notions of SOL is very different from the indeterminacy of the quantifier $Q_0$: In the case of the notions at issue in $\mathscr{L}(Q_0)$ indeterminacy enters through first-order weaknesses of the background theory. The crucial notions of SOL, however, are not only vulnerable to an object-theory level indeterminacy which manifests itself through the possibility of Henkin-models,[51] but, additionally, to an "intermediate" indeterminacy which stems from variability in the meaning of crucial notions, depending on the extent of the underlying set-theoretic universe. The meaning

49. Central notions of SOL are non-absolute, and its Löwenheim-number is much higher than that of FOL, indicating that its involvement with the mathematical background theory is much more substantial. This can also be seen from the meta-theoretical entanglement of SOL with its background set-theory, see (Shapiro 1991) and (Florio and Incurvati 2019).

50. See (Väänänen 1985: 610) and (Väänänen 2021a).

51. See also (Field 1994: 414–420).

of the second-order quantifiers $\forall X$ and $\exists X$ is thus inherently more unstable, and therefore more indeterminate, than that of the quantifier $Q_0$.

Within any particular model of the background set-theory a categoricity theorem ensures that the notions under consideration possess a unique interpretation. Here, the account of §3, as well as the alternatives considered above, succeed in fixing notions required for determinate reference to the natural numbers. Nonetheless, I believe the fact that second-order notions are affected by an additional type of indeterminacy provides substantial reason to prefer a first-order response to the skeptical challenge.

## 5. Concluding Remarks

The response to the sceptical challenge advanced in this paper adopted a richer logical framework, $\mathscr{L}(Q_0)$, that permitted a categorical characterization of $\mathbb{N}$. It was argued that this framework was naturalistically acceptable due to the fact that, in the context of an arithmetical language, it was possible to achieve determinate reference to $Q_0$ in a manner that did not assume infinitary, non-computational powers of deduction. In the course of doing so the proposal advanced the claim that determinate reference to the notions used to formulate a theory does not require recursive axiomatizability.

$\mathscr{L}(Q_0)$ was to be preferred to the more popular approach of using SOL to achieve arithmetical determinacy because indeterminacy runs deeper in the case of SOL than it does in the case of $\mathscr{L}(Q_0)$. Assuming a fixed model of the background theory, the theory of $\mathscr{L}(Q_0)$ suffices to determine a unique value for $Q_0$, whereas the theory of SOL still admits multiple semantics for $\forall X$ and $\exists X$. Less demandingly, assuming a range of admissible models of the background theory of a certain kind does not undermine the determinacy of $Q_0$, whereas the values of the second-order quantifiers might vary depending on the particular model from that class. This suggests that our grasp of $Q_0$ is tighter than our grasp of second-order notions.

The arguments considered here all tried to achieve determinate reference by means of categoricity theorems. Yet, there has been much discussion about the use (and misuse) to which categoricity results have been put in the philosophy of mathematics and logic and several (rather sobering) conclusions about the consequences one may derive from the existence of such results have been drawn.[52] A common complaint against the mathematical realist has been to point out that categoricity theorems of the kind treated above can at most achieve a

52. See (Walmsley 2002), (Meadows 2013), (Button and Walsh 2018), (Maddy and Väänänen 2022).

form of *relative determinacy*, determinacy relative to an assumed interpretation of the background-theory. *That* theory, however, is just as indeterminate as the theory of arithmetic itself was, and its interpretation might be far from unique. Given that any indeterminacy in the background-theory translates back into an indeterminacy of the target-theory, then, it seems not much has been gained by means of a target-theory level categoricity theorem. One reaction to this has been the suggestion to reconceptualize the notion of categoricity with its reliance on a model-theoretic apparatus and replace it with a notion of *internal categoricity*.[53]

The response pursued here was more moderate but, we hope, no less motivated.[54] It rested on the idea that, in order for the challenge of the model-theoretic sceptic to gain traction, he needed to assume at least as much model-theory understood as was needed for convincing the mathematical realist of the existence of non-standard models. The charge of the sceptic was thus that the position of the moderate mathematical realist was self-undermining: given the resources available to her, she was, by her own standards, unable to secure determinate reference, which could be demonstrated using methods accepted by her.[55] This much may thus be assumed when responding to the sceptic: however much set-theory is needed in order to achieve a coherent enough understanding of the notions used in challenging the position of the mathematical realist. And this much suffices, this paper has argued, to defuse the sceptical challenge. What the moderate realist needed to demonstrate to respond to the sceptical challenge was that she had, by her own standards, tools available that sufficed to categorically characterize the natural number structure, that her position was not internally unstable.[56]

This she did by proposing a set of rules that, in the context of arithmetical theories, satisfied the same standards as other rules accepted by her and whose requirements did not go beyond anything not already granted. Further evidence that the acceptance of the rules and the meanings they determined did not go beyond anything not already granted to the moderate realist, that the demands of $\mathscr{L}(Q_0)$ on the set-theoretical background are no higher than the demands of FOL itself, is afforded by a recently suggested assessment of Sagi (2018).[57]

53. See (Parsons 1990; 2007), (Väänänen 2012; 2021b), (Väänänen and Wang 2015), (Button and Walsh 2016; 2018), and (Button 2022).

54. See (Field 2001: 357) for criticism and (Button 2022) for a development of *internal realism*.

55. Cf. (Field 1994: 413) and (Button and Walsh 2018: ch. 9).

56. I am grateful to an anonymous reviewer for asking me to clarify the nature of the challenge to the moderate realist.

57. See also (Kennedy and Väänänen 2021) for discussion of Sagi's suggestion.

Sagi proposes to measure the degree of mathematical involvement of a logic in terms of the *Löwenheim-number* of that logic. The *Löwenheim-number* of a logic, if it exists, is the lowest cardinal $\kappa$ such that if a sentence of the logic has models it has models of cardinality at most $\kappa$. It can be understood as determining the amount of mathematical structure required by the logic, as measuring the strength of the demands on the set-theoretic background, and thus as spelling out the degree of the theory's mathematical involvement. By the downward Löwenheim-Skolem Theorem the Löwenheim-number of FOL is $\aleph_0$. It is known that the Löwenheim-number of $\mathscr{L}(Q_\alpha)$ is $\aleph_\alpha$ for any ordinal $\alpha$[58] and thus, the Löwenheim-number of, in particular, $\mathscr{L}(Q_0)$ is $\aleph_0$. Interpreting the Löwenheim-numbers as measures of the demands placed on the background set-theory it follows that the use of $\mathscr{L}(Q_0)$ requires no more set-theory than the use of FOL.[59]

Within a realist position itself, however, appeal to a categoricity-theorem remains relative: it is categoricity with respect to an assumed background framework. The choice of formal framework to achieve the categoricity result, however, can minimize the dependency on the background and thus the remaining degree of indeterminacy. For the proponent of $\mathscr{L}(Q_0)$ that means, in particular, that as long as especially severe misconstructions of the basic notions of the background set-theory can be avoided, i.e., as long as models of set-theory are assumed to be transitive, she has to be taken to have successfully achieved determinate reference to $\mathbb{N}$. Here, I claimed that approaches in terms of $\mathscr{L}(Q_0)$ fared much better than those in terms of SOL.

What are the prospects for ruling out the remaining indeterminacy and ensuring determinate reference "all the way down"? I think the idea that we can assume the intended model of our background theory must be given up. Yet, as demonstrated above, we can do with something more minimal: we merely need to be certain that the model of the background theory has a particular shape, i.e., is transitive. How reasonable is this assumption? A possible response might go as follows: the radical non-standardness of models that deviantly interpret "set" and "$\in$" among finite sets is something that one might, as Field (1994; 2001) argued, expect to conflict with the application of the mathematical apparatus in practice, and that may therefore be excluded on other grounds. The assumption of the transitivity of the models of the background set-theory is therefore something akin to an idealization of our ability (whatever it may be) to recognize such comparatively basic models as non-standard "by inspection."

58. See (Sagi 2018: 20) and (Väänänen 1985), (Magidor and Väänänen 2011).

59. Löwenheim-numbers, as usually defined, assess the mathematical demands with respect to individual sentences of the logic under consideration. However, the above-mentioned results transfer, in the present case, just as well to entire theories of the respective logics.

## 6. Appendix

We work in the framework of Bonnay and Westerståhl (2016) and thus assume that our first-order language contains, in addition to predicate constants, also predicate variables of any adicity. Without this assumption attention would have to be restricted to *definable* sets (though the difference is minimized by the assumption that quantifiers, as logical operations, must be defined over *all domains*).

Let $\mathscr{L}(Q)$ be the language of FOL with an added quantifier-symbol $Q$. Let $\mathcal{Q}'$ and $\mathcal{Q}^*$ be (type-appropriate) interpretations for $Q$. We say that $\varphi$ is *true in a model* $\mathcal{M}$ *under the* $\mathcal{Q}$*-interpretation*, $\mathcal{M} \vDash^{\mathcal{Q}} \varphi$, if $\mathcal{M} \vDash \varphi$ when $Q$ is interpreted by $\mathcal{Q}$. $\varphi$ *follows from* $\Gamma$ *under the* $\mathcal{Q}$*-interpretation*, $\Gamma \vDash_{\mathcal{Q}} \varphi$ if, for all $\mathcal{M}$, whenever $\mathcal{M} \vDash^{\mathcal{Q}} \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M} \vDash^{\mathcal{Q}} \varphi$. A quantifier $\mathcal{Q}^*$ is *consistent with a consequence relation* $\vDash_{\mathcal{Q}}$ if, for all $\Gamma \cup \{\varphi\}$ and models $\mathcal{M}$, whenever $\Gamma \vDash_{\mathcal{Q}} \varphi$ and $\mathcal{M} \vDash^{\mathcal{Q}^*} \Gamma$ then $\mathcal{M} \vDash^{\mathcal{Q}^*} \varphi$.[60]

Let $\mathcal{Q}_0(M) = \{A \subseteq M \mid \omega \leq |A|\}$ and $\Delta_P = \{\exists_{\geq n} x Px \mid n \in \mathbb{N}\}$ (in general, $\Delta_\varphi = \{\exists_{\geq n} x\varphi(x) \mid n \in \mathbb{N}\}$).

**Theorem 1** *Suppose that*

(i′) $\Delta_P \vDash_{\mathcal{Q}'} QxPx$
(ii′) $QxPx \vDash_{\mathcal{Q}'} \psi$ *for all* $\psi \in \Delta_P$
(i*) $\Delta_P \vDash_{\mathcal{Q}^*} QxPx$
(ii*) $QxPx \vDash_{\mathcal{Q}^*} \psi$ *for all* $\psi \in \Delta_P$

*Then* $\mathcal{Q}' = \mathcal{Q}^*$.

*Proof:* Suppose (i′), (ii′), (i*) and (ii*) hold. Let $\mathcal{M}$ be a model.

"⊆" We will show that $\mathcal{Q}'(M) \subseteq \mathcal{Q}^*(M)$. Let $A \in \mathcal{Q}'(M)$ and set $[\![Px]\!]^{\mathcal{M}} = A$. Then $\mathcal{M} \vDash^{\mathcal{Q}'} QxPx$. Hence, by (ii′), $\mathcal{M} \vDash^{\mathcal{Q}'} \Delta_P$. Since no sentence in $\Delta_P$ contains $Q$, it follows that $\mathcal{M} \vDash^{\mathcal{Q}^*} \Delta_P$ and thus, by (i*), that $\mathcal{M} \vDash^{\mathcal{Q}^*} QxPx$. Hence, $[\![Px]\!]^{\mathcal{M}} = A \in \mathcal{Q}^*(M)$ as desired.

"⊇" Analogous.

Therefore, $\mathcal{Q}' = \mathcal{Q}^*$. □

60. The following results are special cases of results stated and proven in (Speitel 2020) and (Speitel and Westerståhl 2022). They are based on the presentation of the issue in (Bonnay and Westerståhl 2016). The unique determinability of $\mathcal{Q}_0$ was first observed by Dag Westerståhl and is stated and generalized in (Speitel 2020). I am grateful to Gil Sagi for pointing out a mistake in the original proofs.

**Theorem 2** *We have, in particular, the following*

(i) $\Delta_P \vDash_{\mathcal{Q}_0} QxPx$
(ii) $QxPx \vDash_{\mathcal{Q}_0} \psi$ *for all* $\psi \in \Delta_P$

*Proof:* Let $\mathcal{M}$ be a model. Suppose that $\mathcal{M} \vDash \Delta_P$, i.e., that $\mathcal{M} \vDash \exists_{\geq n} xPx$ for all $n \in \mathbb{N}$. Then $[\![Px]\!]^{\mathcal{M}} \geq \omega$ and therefore $[\![Px]\!]^{\mathcal{M}} \in \mathcal{Q}_0(M)$. Hence, $\mathcal{M} \vDash^{\mathcal{Q}_0} QxPx$.

Now suppose that $\mathcal{M} \vDash^{\mathcal{Q}_0} QxPx$. That means that $[\![Px]\!]^{\mathcal{M}} \in \mathcal{Q}_0(M)$ and thus that $[\![Px]\!]^{\mathcal{M}} \geq \omega$. Thus, in particular, $\mathcal{M} \vDash \exists_{\geq n} xPx$ for all $n \in \mathbb{N}$. $\square$

**Corollary 1** *Let* $\mathcal{Q}$ *be a quantifier, such that*

(i) $\Delta_P \vDash_{\mathcal{Q}} QxPx$
(ii) $QxPx \vDash_{\mathcal{Q}} \psi$ *for all* $\psi \in \Delta_P$

*Then* $\mathcal{Q} = \mathcal{Q}_0$.

We will now show how it is possible to replace inference (i) above with a compact inference. Let $R$ be a (new) binary relation symbol and let $LO(R)$ say that $R$ is a left-minimal, right-unbounded, strict linear order; i.e., let $LO(R)$ be the conjunction of the following sentences:

(i) $\forall x \neg xRx$ (*irreflexivity*)
(ii) $\forall x \forall y \forall z (xRy \wedge yRz \rightarrow xRz)$ (*transitivity*)
(iii) $\forall x \forall y (xRy \vee yRx \vee x = y)$ (*connectedness*)
(iv) $\exists x \forall y (x \neq y \rightarrow xRy)$ (*left-minimality*)
(v) $\forall x \exists y\, xRy$ (*right-unboundedness*)

Let $\varphi(x)$ be a formula of FOL with $x$ free and not containing $R$, and let $LO_{\varphi(x)}(R)$ abbreviate that $R$ is a left-minimal, right-unbounded, strict linear order of the $\varphi$'s; i.e., relativize (i)–(v) above to elements satisfying $\varphi(x)$.

**Theorem 3** *Let* $\varphi(x)$ *be a formula of FOL with* $x$ *free and not containing* $R$. *T.f.a.e.:*

(a) $\Delta_\varphi \vDash_{\mathcal{Q}} Qx\varphi(x)$
(b) $LO_{\varphi(x)}(R) \vDash_{\mathcal{Q}} Qx\varphi(x)$

*Proof:*

(a) $\Rightarrow$ (b): Assume that (a) holds and let $\mathcal{M}$ be a model, s.t. $\mathcal{M} \vDash^{\mathcal{Q}} LO_{\varphi(x)}(R)$. Suppose there is an $n$, s.t. $\mathcal{M} \nvDash^{\mathcal{Q}} \exists_{\geq n} x\varphi(x)$. That means, $|[\![\varphi(x)]\!]^{\mathcal{M}}| < n$.

Since the $\varphi$'s form a strict linear order but are finite, it follows that there must be a maximal element. But the order is right-unbounded—contradiction. Thus, there is no $n$, s.t. $\mathcal{M} \nvDash^{Q} \exists_{\geq n} x\varphi(x)$, and therefore $\mathcal{M} \vDash^{Q} \Delta_{\varphi}$. But then, by (a), $\mathcal{M} \vDash^{Q} Qx\varphi(x)$. Therefore, $LO_{\varphi(x)}(R) \vDash_{Q} Qx\varphi(x)$.

(b) $\Rightarrow$ (a): Assume that (b) holds and let $\mathcal{M}$ be a model, s.t. $\mathcal{M} \vDash^{Q} \Delta_{\varphi}$. Since $\Delta_{\varphi}$ is a set of sentences of FOL we know, by the Löwenheim-Skolem Theorem, that there exists a countable elementary submodel $\mathcal{M}^{-}$, $\mathcal{M}^{-} \preccurlyeq \mathcal{M}$, s.t. $\mathcal{M}^{-} \vDash \Delta_{\varphi}$. Since $Q$ does not occur in any sentence of $\Delta_{\varphi}$ we also have that $\mathcal{M}^{-} \vDash^{Q} \Delta_{\varphi}$.
Since $\mathcal{M}^{-} \vDash \Delta_{\varphi}$, $[\![\varphi(x)]\!]^{\mathcal{M}^{-}}$ is infinite. Since $\mathcal{M}^{-}$ is countable, so is $[\![\varphi(x)]\!]^{\mathcal{M}^{-}}$. Enumerate the elements of $[\![\varphi(x)]\!]^{\mathcal{M}^{-}}$ as $a_0, a_1, \ldots$. We define a new relation $\mathcal{R}$ on these elements by setting $\langle a_i, a_j \rangle \in \mathcal{R}$ iff $i < j$. As can easily be checked, $\mathcal{R}$ is a left-minimal, right-unbounded, strict linear ordering of the $\varphi$'s in $\mathcal{M}^{-}$.
Since $\mathcal{M}^{-} \preccurlyeq \mathcal{M}$ we have that $[\![\varphi(x)]\!]^{\mathcal{M}^{-}} \subseteq [\![\varphi(x)]\!]^{\mathcal{M}}$. Now, let $\mathcal{D} = [\![\varphi(x)]\!]^{\mathcal{M}} \backslash [\![\varphi(x)]\!]^{\mathcal{M}^{-}}$ and define $\mathcal{R}^{+} = \mathcal{R} \cup \{\langle a_0, d \rangle, \langle d, a_n \rangle \mid d \in \mathcal{D}; n > 0\}$ and claim that $\mathcal{R}^{+}$ is a strict partial order of the $\varphi$'s in $\mathcal{M}$ (see *proof of claim 1* below).
By the *Order Extension Principle* there then exists a strict linear order $\mathcal{R}^{*}$ of the $\varphi$'s in $\mathcal{M}$, s.t. $\mathcal{R}^{+} \subseteq \mathcal{R}^{*}$. We claim that $\mathcal{R}^{*}$ is, in addition, left-minimal and right-unbounded (see *proof of claim 2* below).
Now let $\mathcal{R}^{*}$ interpret $R$. Then, $\langle \mathcal{M}, \mathcal{R}^{*} \rangle \vDash^{Q} LO_{\varphi(x)}(R)$ and thus, by (b), $\langle \mathcal{M}, \mathcal{R}^{*} \rangle \vDash^{Q} Qx\varphi(x)$. Since $R$ does not occur anywhere in $Qx\varphi(x)$ it follows that $\mathcal{M} \vDash^{Q} Qx\varphi(x)$ as well. □

*Proof of claim 1*: $\mathcal{R}^{+}$ is a strict partial order of the $\varphi$'s in $\mathcal{M}$.

(a) *Irreflexivity*: Let $b \in [\![\varphi(x)]\!]^{\mathcal{M}}$. Then either (i) $b \in [\![\varphi(x)]\!]^{\mathcal{M}^{-}}$ or (ii) $b \in \mathcal{D}$. Note that $d \neq a_n$ for any $d \in \mathcal{D}$ and $n \in \mathbb{N}$.
If (i) and $\langle b, b \rangle \in \mathcal{R}^{+}$, then already $\langle b, b \rangle \in \mathcal{R}$. But $\mathcal{R}$ was irreflexive and thus $\langle b, b \rangle \notin \mathcal{R}$. Hence, $\langle b, b \rangle \notin \mathcal{R}^{+}$.
If (ii), $\langle b, b \rangle \notin \mathcal{R}^{+}$ immediately follows, since $\mathcal{R}^{+}$ contains no tuples both components of which are members of $\mathcal{D}$.
Hence, $\mathcal{R}^{+}$ is irreflexive.

(b) *Transitivity*: Let $a, b, c \in [\![\varphi(x)]\!]^{\mathcal{M}}$ and suppose that $\langle a, b \rangle, \langle b, c \rangle \in \mathcal{R}^{+}$. Note that, by (a), $a \neq b$ and $b \neq c$.

(i) *Case 1: $a, b, c \notin \mathcal{D}$.*
It follows that $\langle a, b \rangle, \langle b, c \rangle \in \mathcal{R}$. Since $\mathcal{R}$ is transitive we have that $\langle a, c \rangle \in \mathcal{R}$ as well, and thus that $\langle a, c \rangle \in \mathcal{R}^{+}$.

(ii) *Case 2: $a \in \mathcal{D}$; $b, c \notin \mathcal{D}$.*
Since $\langle b, c\rangle \in \mathcal{R}^+$ it follows that $c = a_n$ for some $n > 0$. Hence, by the construction of $\mathcal{R}^+$, $\langle a, c\rangle \in \mathcal{R}^+$.

(iii) *Case 3: $b \in \mathcal{D}$; $a, c \notin \mathcal{D}$.*
Since $\langle a, b\rangle \in \mathcal{R}^+$ we have that $a = a_0$. Since $\langle b, c\rangle \in \mathcal{R}^+$ we have that $c = a_n$ for some $n > 0$. Hence, $\langle a, c\rangle \in \mathcal{R}$ and thus also $\langle a, c\rangle \in \mathcal{R}^+$.

(iv) *Case 4: $c \in \mathcal{D}$; $a, b \notin \mathcal{D}$.*
Since $\langle b, c\rangle \in \mathcal{R}^+$ we have that $b = a_0$. Since there is no $e$, s.t. $\langle e, a_0\rangle \in \mathcal{R}^+$, but $\langle a, b\rangle \in \mathcal{R}^+$ this case is impossible.

(v) *Case 5: $a, b \in \mathcal{D}$; $c \notin \mathcal{D}$.*
Impossible, since $\mathcal{R}^+$ contains no tuples both components of which are members of $\mathcal{D}$.

(vi) *Case 6: $a, c \in \mathcal{D}$; $b \notin \mathcal{D}$.*
In this case it must be the case that $b = a_0$. But there is no $e$, s.t. $\langle e, a_0\rangle \in \mathcal{R}^+$. Hence, this case is impossible.

(vii) *Case 7: $b, c \in \mathcal{D}$; $a \notin \mathcal{D}$.*
Impossible, since $\mathcal{R}^+$ contains no tuples both components of which are members of $\mathcal{D}$.

(viii) *Case 8: $a, b, c \in \mathcal{D}$.*
Impossible, since $\mathcal{R}^+$ contains no tuples both components of which are members of $\mathcal{D}$.

Hence, $\mathcal{R}^+$ is transitive. It follows that $\mathcal{R}^+$ is a strict partial order.

*Proof of claim 2*: $\mathcal{R}^*$ is left-minimal and right-unbounded.

(a) *Left-minimality*: Since $\mathcal{R}^+ \subseteq \mathcal{R}^*$ we know that $\langle a_0, b\rangle \in \mathcal{R}^*$ for all $b \in [\![\varphi(x)]\!]^{\mathcal{M}}$, s.t. $b \neq a_0$. Thus, the left-minimality of $\mathcal{R}^*$ immediately follows from the fact that $\mathcal{R}^*$ is a strict linear order.

(b) *Right-unboundedness*: Take any $b \in [\![\varphi(x)]\!]^{\mathcal{M}}$. Then there exists an $a_n$, s.t. $\langle b, a_n\rangle \in \mathcal{R}^+$. Since $\mathcal{R}^+ \subseteq \mathcal{R}^*$ it follows that $\langle b, a_n\rangle \in \mathcal{R}^*$ as well, and thus that $\mathcal{R}^*$ is right-unbounded.

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