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Estimates of the Bounds of $\pi(x)$ and $\pi((x+1)^2) - \pi(x^2)$

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We show the following bounds on the prime counting function $\pi(x)$ using principles from analytic number theory, giving an estimate

$$\log 2 \le \liminf_{x\to\infty} \frac{\pi(x)}{x/\log x} \le \limsup_{x\to\infty} \frac{\pi(x)}{x/\log x} \le 2\log 2$$

We also conjecture about the bounding of $\pi((x+1)^2) - \pi(x^2)$, as is relevant to Legendre's conjecture about the number of primes in the aforementioned interval such that:

$$\left\lfloor \frac{1}{2}\left(\frac{(x+1)^2}{\log(x+1)} - \frac{x^2}{\log x}\right) - \frac{(\log x)^2}{\log(\log x)} \right\rfloor \le \pi\left((x+1)^2\right) - \pi\left(x^2\right) \le \left\lfloor \frac{1}{2}\left(\frac{(x+1)^2}{\log(x+1)} - \frac{x^2}{\log x}\right) - \log^2 x \log\log x \right\rfloor$$

1. Introduction

Recall the definition:

$$\pi(x) := \sum_{\substack{p\le x\\ p\,\text{prime}}} 1,$$

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and let us define the following:

$$\Theta(x) := \sum_{\substack{p \leq x \\ p \text{ prime}}} \log p,$$

$$\Psi(x) := \sum_{1 \leq n \leq x} \Lambda(n) = \sum_{\substack{p^m \leq x \\ m \geq 1 \\ p \text{ prime}}} \log p,$$

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime } p \text{ and integer } k \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

The following are simple statements from real analysis that are required for rigorousness' sake: let $\{x_n\}$ be a sequence of real numbers and L be a real number with the following two properties: $\forall \epsilon > 0$, $\exists N$ such that $x_n < L + \epsilon$, $\forall n \geq N$. $\forall \epsilon > 0$ $\wedge$ $N \geq 1$, $\exists n \geq N$ with $x_n > L - \epsilon$. We thus define L as:

$$\limsup_{n \to \infty} x_n = L$$

Thus on the contrary we must have:

$$\liminf_{n \to \infty} x_n = -\limsup_{n \to \infty} -x_n$$

2. Necessary Preliminary Results

Theorem 2.1. *For all* $\alpha \in (0, 1)$, *and all* $x \geq x_0$:

$$\frac{\Theta(x)}{\log(x)} \leq \frac{\Psi(x)}{\log(x)} \leq \pi(x) \leq \frac{\Theta(x)}{\alpha \log(x)} + x^{\alpha}$$

Proof. Clearly $\Theta(x) \leq \Psi(x)$, such that

$$\limsup_{x \to \infty} \frac{\Psi(x)}{x} \geq \limsup_{x \to \infty} \frac{\Theta(x)}{x}$$

Also, if p is a prime and $p^m \leq x < p^{m+1}$, then $\log p$ occurs in the sum for $\Psi(x)$ exactly m times. [1]

$$\begin{aligned}\Psi(x) &= \sum_{\substack{p^m \le x \\ p \text{ prime} \\ m \ge 1}} \log p \\ &= \sum_{\substack{p \le x \\ p \text{ prime}}} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p \\ &\le \sum_{\substack{p \le x \\ p \text{ prime}}} \log x \\ &= \pi(x) \log x\end{aligned}$$

$$\limsup_{x\to\infty} \frac{\Psi(x)}{x} \le \limsup_{x\to\infty} \frac{\pi(x)}{x/\log x}$$

Now fix $\alpha \in (0, 1)$. Given $x > 1$,

$$\Theta(x) = \sum_{\substack{p \le x \\ p \text{ prime}}} \log p \ge \sum_{\substack{x^\alpha < p \le x \\ p \text{ prime}}} \log p .$$

It is clear that all p from the second sum satisfy: $\log p > \alpha \log x$.

$\therefore$

$$\begin{aligned}\Theta(x) &> \alpha \log x \sum_{\substack{x^\alpha < p \le x \\ p \text{ prime}}} 1 \\ &= \alpha \log x \left(\pi(x) - \pi(x^\alpha)\right) \\ &> \alpha \log x \left(\pi(x) - x^\alpha\right)\end{aligned}$$

$\ni$

$$\frac{\Theta(x)}{x} > \frac{\alpha \pi(x)}{x/\log x} - \frac{\alpha \log x}{x^{1-\alpha}}$$

$\forall \alpha \in (0, 1)$ we have:

$$\lim_{x\to\infty} \frac{\alpha \log x}{x^{1-\alpha}} = 0.$$

Combining these we get:

$$\limsup_{x\to\infty}\frac{\Theta(x)}{x}\geq\limsup_{x\to\infty}\frac{\alpha\pi(x)}{x\,/\log x}$$

Once again, since our statement is true $\forall\alpha\in(0,1)$,

$$\limsup_{x\to\infty}\frac{\Theta(x)}{x}\geq\limsup_{x\to\infty}\frac{\pi(x)}{x\,/\log x}$$

Similarly:

$$\liminf_{x\to\infty}\frac{\Psi(x)}{x}\geq\liminf_{x\to\infty}\frac{\Theta(x)}{x}$$

$$\liminf_{x\to\infty}\frac{\pi(x)}{x/\log x}\geq\liminf_{x\to\infty}\frac{\Psi(x)}{x}$$

Once again, we apply the same method:

$$\liminf_{x\to\infty}\frac{\Theta(x)}{x}\geq\liminf_{x\to\infty}\frac{\pi(x)}{x/\log x},$$

and have thus proven **Theorem 2.1**. □

3. Main Result

Theorem 3.1. *We have:*

$$\log 2\leq\liminf_{x\to\infty}\frac{\pi(x)}{x/\log x}\leq\limsup_{x\to\infty}\frac{\pi(x)}{x/\log x}\leq 2\log 2.$$

Proof. First the lower bound. Take:

$$S(x):=\sum_{1\leq n\leq x}\log n-2\sum_{1\leq n\leq x/2}\log n.$$

Then,

$$\begin{aligned}\sum_{\substack{1\le d\le n\\ d|n}}\Lambda(d) &= \sum_{\substack{p^r|d\\ d|n\\ p\,\text{prime}}}\log p\\ &= \sum_{i=1}^{l}\sum_{r=1}^{e_i}\log p_i \quad \text{where } n = p_1^{e_1}\cdots p_l^{e_l}\\ &= \sum_{i=1}^{l} e_i \log p_i\\ &= \log n\end{aligned}$$

∴

$$S(x) = \sum_{1\le n\le x}\sum_{d|n}\Lambda(d) - 2\sum_{1\le n\le x/2}\sum_{d|n}\Lambda(d)$$

Clearly $\{d, 2d, \ldots, qd\}$ is the set of n satisfying $1 \le n \le x$ and $d \mid n$ (we can see this easily by writing $x = r + qd$ with $0 \le r < d$).

∴

$$\begin{aligned}S(x) &= \sum_{1\le d\le x}\Lambda(d)\left[\frac{x}{d}\right] - 2\sum_{1\le d\le x/2}\Lambda(d)\left[\frac{x}{2d}\right]\\ &= \sum_{1\le d\le x/2}\Lambda(d)\left(\left[\frac{x}{d}\right] - 2\left[\frac{x}{2d}\right]\right) + \sum_{(x/2)<d\le x}\Lambda(d)\left[\frac{x}{d}\right]\\ &\le \sum_{1\le d\le x/2}\Lambda(d) + \sum_{(x/2)<d\le x}\Lambda(d)\\ &= \Psi(x).\end{aligned}$$

So,

$$\frac{\Psi(x)}{x} \ge \frac{S(x)}{x} = \frac{1}{x}\sum_{1\le n\le x}\log n - \frac{2}{x}\sum_{1\le n\le x/2}\log n.$$

$\log t$ is increasing,

∴

$$\int_1^{x+1}\log t\,dt \ge \sum_{1\le n\le x}\log n,$$

$$\int_1^{[x]}\log t\,dt \le \sum_{1\le n\le x}\log n.$$

Actually, assuming $x \in \mathbb{Z}^+$,

$$\begin{aligned}
\frac{S(x)}{x} &\geq \frac{1}{x}\int_1^x \log t\, dt - \frac{2}{x}\int_1^{(x/2)+1} \log t\, dt \\
&= \frac{1}{x}\left(x\log x - x + 1\right) - \frac{2}{x}\left(\frac{x+2}{2}\log\left(\frac{x+2}{2}\right) - \frac{x+2}{2} + 1\right) \\
&= \log x + \frac{1}{x} - \frac{x+2}{x}\log(x+2) + \frac{x+2}{x}\log 2 \\
&> \log\left(\frac{x}{x+2}\right) - \frac{2}{x}\log(x+2) + \log 2
\end{aligned}$$

Using **Theorem 2.1**, we get:

$$\begin{aligned}
\liminf_{x\to\infty} \frac{\pi(x)}{x/\log x} &= \liminf_{x\to\infty} \frac{\Psi(x)}{x} \\
&\geq \liminf_{x\to\infty} \frac{S(x)}{x} \\
&> \lim_{x\to\infty} \log\left(x/(x+2)\right) - \frac{2}{x}\log(x+2) + \log 2 \\
&= \log 2
\end{aligned}$$

To complete the proof, we will need some auxiliary results taken from Murty's *Analytic Number Theory* [1] in the form of three lemmas:

Lemma 3.2.

$$\operatorname{ord}_p(m!) = \sum_{r\geq 1}\left[\frac{m}{p^r}\right], \forall m \in \mathbb{Z}^+,\ \textit{prime}\ p$$

Proof. Fix an exponent r. The positive integers no larger than m that are multiples of p^r are

$$p^r, 2p^r, \ldots, \left[\frac{m}{p^r}\right]p^r$$

and those that are multiples of p^{r+1} are

$$p^{r+1}, 2p^{r+1}, \ldots, \left[\frac{m}{p^{r+1}}\right]p^{r+1}$$

Thus there are precisely $\left[m/p^r\right]-\left[m/p^{r+1}\right]$ positive integers $n \le m$ with $\mathrm{ord}_p(n) = r$.

$\therefore$

$$\begin{aligned}\mathrm{ord}_p(m!) &= \sum_{n=1}^{m} \mathrm{ord}_p(n) \\ &= \sum_{r\ge1} \sum_{\substack{1\le n\le m \\ \mathrm{ord}_p(n)=r}} r \\ &= \sum_{r\ge1} r\left(\left[m/p^r\right]-\left[m/p^{r+1}\right]\right) \\ &= \sum_{r\ge1} r\left[m/p^r\right]-\sum_{r\ge1} r\left[m/p^{r+1}\right] \\ &= \sum_{r\ge1} r\left[m/p^r\right]-\sum_{r\ge1} (r-1)\left[m/p^r\right] \\ &= \sum_{r\ge1}\left[\frac{m}{p^r}\right]\end{aligned}$$

□

Lemma 3.3. $\forall n \in \mathbb{Z}^+$,

$$\frac{2^{2n}}{2\sqrt{n}} < \binom{2n}{n} < \frac{2^{2n}}{\sqrt{2n+1}}$$

Proof.

$$\begin{aligned}P_n &:= \prod_{i\le n} \frac{(2i-1)}{(2i)} \\ &= \frac{(2n)!}{2^{2n}(n!)^2} \\ &= \binom{2n}{2}\frac{1}{2^{2n}}\end{aligned}$$

Since:

$$\frac{(2i-1)(2i+1)}{(2i)^2} < 1$$

for all $i \ge 1$.

$\therefore$

$$1 > (2n+1)P_n^2,$$

giving the upper bound. For the lower bound:

$$1-\frac{1}{(2i-1)^2}<1$$

$\forall i \geq 1$, such that

$$1>\prod_{i=2}^{n}\left(1-\frac{1}{(2i-1)^2}\right)$$

$$=\prod_{i=2}^{n}\frac{(2i-1)^2-1}{(2i-1)^2}$$

$$=\prod_{i=2}^{n}\frac{(2i-2)(2i)}{(2i-1)^2}$$

$$=\frac{1}{4nP_n^2}$$

yielding our lemma. □

Lemma 3.4. $\forall n \in \mathbb{Z}^+$,

$$\Theta(n)<2n\log 2$$

Proof. By **Lemma 3.3**,

$$\log\left(\binom{2n}{n}\frac{1}{2}\right)=\log\left(\binom{2n}{n}\right)-\log 2$$

$$<2n\log 2-\frac{1}{2}\log(2n+1)-\log 2$$

$$=(2n-1)\log 2-\frac{1}{2}\log(2n+1)$$

since

$$\binom{2n}{n}\frac{1}{2}=\frac{(2n)!}{(n!)^2}\frac{n}{2n}=\frac{(2n-1)!}{n!(n-1)!}=\binom{2n-1}{n-1}.$$

by **Lemma 3.2**:

$$\log\left(\binom{2n}{n}\frac{1}{2}\right)=\log\binom{2n-1}{n-1}$$

$$= \sum_{p \text{ prime}} \operatorname{ord}_p\left((2n-1)!\right)\log p - \sum_{p \text{ prime}} \operatorname{ord}_p\left((n-1)!\right)\log p - \sum_{p \text{ prime}} \operatorname{ord}_p\left(n!\right)\log p$$

$$= \sum_{p \text{ prime}} \log p \sum_{r\ge 1}\left[(2n-1)/p^r\right]-\left[n/p^r\right]$$

$$\ge \sum_{\substack{p \text{ prime}\\ n<p\le 2n-1}} \log p$$

$$= \Theta(2n-1)-\Theta(n)$$

Where

$$\Theta(2n-1)-\Theta(n)<(2n-1)\log 2-\frac{1}{2}\log(2n+1)$$

We now proceed by induction. Proceeding from the trivialities, suppose $m>2$ and the lemma is true for $n<m, n, m \in \mathbb{N}$. If m is odd, then $m = 2n - 1$ for some integer $n \ge 2$ since $m > 2$. Thus by induction,

$$\Theta(m) = \Theta(2n-1) < \Theta(n) + (2n-1)\log 2 - \frac{1}{2}\log(2n+1)$$

$$< 2n\log 2 + (2n-1)\log 2 - \frac{1}{2}\log(2n)$$

$$= (4n-1)\log 2 - \frac{1}{2}\log(2n)$$

$$\le (4n-2)\log 2 \quad (\textit{since}\, n \ge 2)$$

$$= 2m\log 2$$

If m is even, then $m = 2n$ for some integer n with $m > n \ge 2$ and m is composite. Clearly $\Theta(m) = \Theta(m-1)$ and we know:

$$\Theta(m) = \Theta(m-1) < 2(m-1)\log 2 < 2m\log 2$$

Lemma 3.4 gives

$$2\log 2 \geq \limsup_{x\to\infty} \frac{\Theta(x)}{x}. \qquad \square$$

The desired lower bound follows from **Theorem 2.1** $\square$

4. On Primes in the Gaps between Squares

The following is relatively aleatory compared to the previous workings, but it is worth mentioning considering the importance of the statement.

By Hassani [2], we have

$$\left\lfloor \frac{1}{2}\left(\frac{(x+1)^2}{\log(x+1)} - \frac{x^2}{\log x}\right) - \frac{\log^2 x}{\log\log x} \right\rfloor \leq \pi\left((x+1)^2\right) - \pi\left(x^2\right)$$

$$\frac{1}{2}\left(\frac{x^2}{\log n} - \frac{3^2}{\log 3}\right) - \sum_{j=3}^{x-1} \frac{\log^2 x}{\log\log k} < \pi\left(x^2\right) - \pi\left(3^2\right)$$

And thus:

$$\sum_{j=3}^{x-1} \left\lfloor \frac{1}{2}\left(\frac{(j+1)^2}{\log(j+1)} - \frac{j^2}{\log j}\right) - \frac{\log^2 j}{\log\log j} \right\rfloor < \sum_{j=3}^{n-1} \pi\left((j+1)^2\right) - \pi\left(j^2\right)$$

$\therefore$

$$\left\lfloor \frac{1}{2}\left(\frac{(x+1)^2}{\log(x+1)} - \frac{x^2}{\log x}\right) - \frac{(\log x)^2}{\log(\log x)} \right\rfloor \leq \pi\left((x+1)^2\right) - \pi\left(x^2\right).$$

And by the prime number theorem, which gives us the asymptotic estimate for some

$$F(x) := \pi\left((x+1)^2\right) - \pi\left(x^2\right) \sim \frac{1}{2}\left(\frac{(x+1)^2}{\log(x+1)} - \frac{x^2}{\log x}\right)$$

We propose:

$$\pi\left((x+1)^2\right) - \pi\left(x^2\right) \leq \left\lfloor \frac{1}{2}\left(\frac{(x+1)^2}{\log(x+1)} - \frac{x^2}{\log x}\right) + \log^2 x \log\log x \right\rfloor$$

by the same method.

References

1. R. Ram Murty, *Problems in Analytic Number Theory, Second Edition — Graduate Texts in Mathematics*, Springer, 2001.
2. M. Hassani, *Counting primes in the interval* (n^2, $(n+1)^2$), AMS, 1997.